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A LEMMA ON MATRICES AND A CONSTRUCTION OF MULTI-WAVELETS

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Dedicated to Professor Rémi Vaillancourt on the occasion of his sixtieth birthday

Abstract. A generalization of Gröchenig's lemma on matrices is given. A theory of multi-dimensional r-regular multi-wavelets is described in general terms. A general existence theorem for multi-dimensional r-regular multi-wavelets based on the generalization of Gröchenig's lemma, similar to the general existence theorem for Meyer's r-regular wavelets, is proved.

1. Introduction

Wavelets have their origin in many fields of pure and applied mathematics. It is usual for wavelets to be generated by a scaling function. On the contrary, multi-wavelets are generated by many scaling functions, which gives us advantage. It is believed that multi-wavelets are ideally suited to multichannel signals like color images which are two-dimensional three-channel signals and stereo audio signals which are one-dimensional two-channel signals.

Many papers deal with multi-wavelets and various constructions of multi-wavelets are already known. For example, Alpert [1] generalized the Haar system to one-dimensional non-regular multi-wavelets in $L^2(\mathbb{R}^1)$ having vanishing moments by producing an example of such multi-wavelets. In [17], Strang and Strela constructed a pair of real-valued one-dimensional multi-wavelets with short support and symmetry, and, in [18], they constructed a nonsymmetric pair; both of these cases are in $L^2(\mathbb{R}^1)$. Jia and Shen [13] investigated multiresolution on the basis of shift-invariant spaces, proved a general existence theorem and gave examples to illustrate the general theory. Their constructions are different from ours.

We shall generalize Gröchenig's lemma on matrices [8], introduce n-dimensional r-regular multi-wavelets in $L^2(\mathbb{R}^n)$, and give a general existence theorem, which follows the framework of Meyer's general existence theorem [15, Theorem 2 of Section 3.6 and Proposition 4 of Section 3.7] for r-regular wavelets which, in this paper, will be called r-regular *single-wavelets*.

In Section 2, we shall give a genaralization of Gröchenig's lemma on matrices, Theorem 1, and introduce an r-regular multiresolution analysis for multi-dimensional multi-wavelets and state Theorems 2 and 3. Our main results are Theorem 1 and Theorem 3, which is a general existence theorem for multi-dimensional r-regular multi-wavelets asserting that the existence of an r-regular multi-wavelets is reduced to the existence of an r-regular multiresolution analysis. Our definition of a multiresolution analysis for multi-wavelets needs a stronger assumption than that of Meyer for single-wavelets. We say nothing on the

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existence of an r-regular multiresolution analysis in general. We shall give an example of an r-regular multiresolution analysis for multi-wavelets using a $(2r+1)$-regular multiresolution analysis for single-wavelets.

In Section 3, we shall prove Theorem 1.

In Section 4, we shall give a brief introduction to basic properties of multi-wavelets leading to Proposition 4, which is the basis of the existence results of this paper. Though this is well-known procedure leading to a construction of multi-wavelets, we stress the r-regularity.

In Section 5, we shall give the proofs of Theorems 2 and 3 using Theorem 1. These theorems give a different construction of r-regular multi-wavelets from known constructions.

2. Definition and Main Results

First, we start with Theorem 1, which is a generalization of Gröchenig's lemma on matrices.

We assume that every manifold satisfies the second axiom of countability, that is, it has a countable basis of open sets. We denote by (z, w) the standard Hermitian product of $z = (z_i)$ and $w = (w_i)$ in $\mathbb{C}^m$.

Theorem 1. *Let X be a real, compact, C^∞-manifold with $\dim X = n$, and let m, n and d be positive integers satisfying*

$$2 \leq 2d \leq 2m - n. \tag{2.1}$$

Then, for all C^∞-mappings $f_\ell : X \longrightarrow \mathbb{C}^m$, $\quad \ell = 1, \dots, d$, with the property

$$(f_k(x), f_\ell(x)) = \delta_{k\ell}, \qquad \text{for } k, \ell \in \{1, \dots, d\},\ x \in X, \tag{2.2}$$

there exist C^∞-mappings $f_\ell : X \longrightarrow \mathbb{C}^m$, $\quad \ell = d+1, \dots, m$, with the property

$$(f_k(x), f_\ell(x)) = \delta_{k\ell}, \qquad \text{for } k, \ell \in \{1, \dots, m\},\ x \in X. \tag{2.3}$$

Remark 1. Let $\{e_j\}_{j=1,\dots,m}$ be the standard basis of $\mathbb{C}^m$. For the mappings $f_\ell : X \longrightarrow \mathbb{C}^m$, $\ell = 1, \dots, m$, put $f_{j\ell}(x) := (f_\ell(x), e_j)$, $\quad j, \ell = 1, \dots, m$. Then (2.3) is equivalent to the fact that the matrix $(f_{j\ell}(x);\ j \downarrow 1, \dots, m,\ \ell \to 1, \dots, m)$ is unitary for each $x \in X$.

Remark 2. In our application of Theorem 1 to the construction of multi-wavelets, we take $X = \mathbb{T}^n$ and $m = 2^n d$. In this case, the inequalities (2.1) are valid for each $n \in \mathbb{N}$ and each $d \geq 1$. Indeed, from the inequality $2^{n+1} \geq n + 2$, $\ n \in \mathbb{N}$, we obtain

$$2m - 2d = (2^{n+1} - 2)d \geq 2^{n+1} - 2 \geq n.$$

Next we give notation and definitions of multi-dimensional multi-wavelets.

Notation 1. The following notation will be used.

- $f_{jk}(x)$ is the scaled and shifted function

$$f_{jk}(x) = 2^{nj/2} f(2^j x - k), \qquad j \in \mathbb{Z},\ k \in \mathbb{Z}^n, \qquad f \in L^2(\mathbb{R}^n). \tag{2.4}$$

- F_{jk} is the vector of scaled and shifted functions

$$F_{jk} = ((f_1)_{jk}, \dots, (f_d)_{jk}), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^n, \quad F = (f_1, \dots, f_d) \in L^2(\mathbb{R}^n)^d.$$

- $R = \{0,1\}^n$ is the set of 2^n vertices of the n-dimensional unit cube.
- $E = R \setminus \{(0,\dots,0)\}$ is the set of vertices of R less the origin.
- $D = \{1,\dots,d\}$ for a positive integer d.
- $\mathbb{N} = \{0,1,2,\dots\}$ is the set of natural numbers including zero.
- $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \simeq [0,2\pi[$ is the one-dimensional torus.
- $2\mathbb{T} = \mathbb{R}/\pi\mathbb{Z} \simeq [0,\pi[$.
- $r \in \mathbb{N}$ throughout the paper.
- $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n), \alpha_j \in \mathbb{N}$, is a multi-index of nonnegative integers.
- $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ is the length of the multi-index α.
- $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$.
- $m(\xi)$, with $\xi \in \mathbb{R}^n$, is $2\pi\mathbb{Z}^n$-periodic if it is 2π-periodic in each ξ_j, $j = 1,2,\dots,n$, that is, $m(\xi)$ is a function on $\mathbb{T}^n$.
- $U(n)$, $n \in \mathbb{N}\setminus\{0\}$, is the unitary group of order n, that is, the group of $n\times n$ unitary matrices.

Definition 1. A family $\{\Psi_\varepsilon\}_{\varepsilon\in E}$ is called a family of 2^n-1 *multi-wavelet*, or *wavelet*, *functions* $\Psi_\varepsilon := (\psi_{\varepsilon 1},\dots,\psi_{\varepsilon d}) \in L^2(\mathbb{R}^n)^d$ if $\{(\psi_{\varepsilon\delta})_{jk}(x) := 2^{nj/2}\psi_{\varepsilon\delta}(2^j x - k)\}_{\varepsilon\in E,\delta\in D,j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. The $(\psi_{\varepsilon\delta})_{jk}$ are called *multi-wavelets*.

Remark 3. An intuitive geometric explanation why $2^n - 1$ wavelet functions are needed is as follows. If, after approximating $\mathbb{R}^n$ by the lattice $\mathbb{Z}^n$, we want to approximate it by the more refined lattice $\frac{1}{2}\mathbb{Z}^n$, then we need to add 2^n-1 extra points for every point in $\mathbb{Z}^n$. We use a function in $L^2(\mathbb{R}^n)^d$ to approximate every lattice point. A functional analytic answer will be given in Remark 9 below.

Definition 2. A family of wavelet functions $\{\Psi_\varepsilon\}_{\varepsilon\in E}$ is said to be *r-regular* if every $\psi_{\varepsilon\delta}$ satisfies the following three conditions.

(c1) **Regularity:**

$$\psi_{\varepsilon\delta}^{(\alpha)}(x) := \partial_x^\alpha \psi_{\varepsilon\delta}(x) \in L^\infty(\mathbb{R}^n), \qquad \varepsilon\in E,\ \delta\in D,\ |\alpha|\le r. \tag{2.5}$$

(c2) **Localization:** For every positive number N, there exists a positive number C_N such that

$$|\psi_{\varepsilon\delta}^{(\alpha)}(x)| \le C_N(1+|x|)^{-N}, \quad \text{a.a. } x, \qquad \varepsilon\in E,\ \delta\in D,\ |\alpha|\le r. \tag{2.6}$$

(c3) **Oscillation:**

$$\int_{\mathbb{R}^n} x^\alpha \psi_{\varepsilon\delta}(x)\,dx = 0, \qquad \varepsilon\in E,\ \delta\in D,\ |\alpha|\le r. \tag{2.7}$$

Remark 4. Condition (c3) is equivalent to $\widehat{\psi}_{\varepsilon\delta}^{(\alpha)}(0) = 0$, for $|\alpha|\le r$, where $\psi(x)$ and its Fourier transform $\widehat{\psi}(\xi)$ are related by the formulae

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\psi(x)\,dx, \qquad \psi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\psi}(\xi)\,d\xi.$$

A central feature of wavelets is their localizing property in both the x- and ξ-spaces. Since the support of $(\psi_{\varepsilon\delta})_{jk}$ becomes very big as $j\to-\infty$, even if every $(\psi_{\varepsilon\delta})_{jk}$ has compact support, we look for appropriate bases for the spaces spanned by $\{(\psi_{\varepsilon\delta})_{jk}\}_{\varepsilon\in E,\delta\in D,j\in\{-1,-2,\dots\},k\in\mathbb{Z}^n}$ by considering the following closed subspaces of $L^2(\mathbb{R}^n)$.

Notation 2. For all $j \in \mathbb{Z}$, let

$$W_{j\delta} := \overline{\mathrm{Span}\{(\psi_{\varepsilon\delta})_{jk}\}}_{\varepsilon\in E, k\in\mathbb{Z}^n}, \quad \delta \in D; \tag{2.8}$$

$$V_{j\delta} := \bigoplus_{k=-\infty}^{j-1} W_{k\delta}, \quad \delta \in D; \qquad W_j := \bigoplus_{\delta\in D} W_{j\delta}; \qquad V_j := \bigoplus_{k=-\infty}^{j-1} W_k.$$

Definition 3. A function $\Phi := {}^t(\varphi_1, \dots, \varphi_d) \in (V_0)^d$ is called a *multi-scaling*, or *scaling function* if $\{\varphi_\delta(x-k)\}_{\delta\in D, k\in\mathbb{Z}^n}$ is an orthonormal basis of V_0. The scaling function $\Phi(x)$ is said to be *r-regular* if it satisfies the above regularity and localization conditions (c1) and (c2) and the following oscillation condition:

(c4) **Oscillation:**

$$\int_{\mathbb{R}^n} x^\alpha \varphi_\delta(x)\, dx = 0, \qquad \delta \in D,\ 1 \le |\alpha| \le 2r+1. \tag{2.9}$$

Remark 5. Lemma 9 will show that the condition $\sum_{\delta\in D} |\widehat{\varphi}_\delta(0)|^2 = 1$ is necessary for the existence of an r-regular scaling function. Hence there exists $\delta \in D$ such that $\int \varphi_\delta(x)\, dx \neq 0$. In the case of single-wavelets, as Meyer stated in [15, Section 2.10, Proposition 7], $\int \varphi(x)\, dx \neq 0$ implies $\int x^\alpha \varphi(x)\, dx = 0$, $1 \le |\alpha| \le 2r+1$ by changing $\varphi(x)$ suitably. But, in the case of multi-wavelets, this implication is still open. We shall show that (c4) implies (c3) using a similar framework to Meyer's. In Daubechies' framework [4, Section 5.5], it is known that the regularities and the localization properties of a wavelet function and the orthonormality of wavelets imply (c3) without (c4).

Definition 4. The generalized Fourier series expansion with respect to the orthonormal basis $\{(\psi_{\varepsilon\delta})_{jk}\}_{\varepsilon\in E,\delta\in D, j\in\mathbb{N}, k\in\mathbb{Z}^n} \cup \{\varphi_\delta(x-k)\}_{\delta\in D, k\in\mathbb{Z}^n}$ is called a *multi-wavelet expansion*.

Remark 6. By Definition 4, $\{(\psi_{\varepsilon\delta})_{jk}\}_{\varepsilon\in E,\delta\in D,k\in\mathbb{Z}^n} \cup \{(\varphi_\delta)_{jk}\}_{\delta\in D,k\in\mathbb{Z}^n}$ is an orthonormal basis of V_{j+1} for every $j \in \mathbb{Z}$.

To construct r-regular wavelet and scaling functions, we use a *multiresolution analysis* [15], by which wavelet functions can be constructed from a given scaling function, $\Phi(x)$.

Definition 5. An increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$,

$$\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots,$$

is called a *multiresolution analysis* if it satisfies the following four properties:

(a) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$;
(b) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$;
(c) $f(x) \in V_0$ if and only if $f(x-k) \in V_0$ for every $k \in \mathbb{Z}^n$;
(d) there exists a function $\Phi(x) := {}^t(\varphi_1(x), \dots, \varphi_d(x)) \in (V_0)^d$ such that $\{\varphi_\delta(x-k)\}_{\delta\in D,k\in\mathbb{Z}^n}$ forms an orthonormal basis of V_0.

Definition 6. A sequence of functions $\{g_k\}_{k\in\mathbb{Z}^n}$ is called a *Riesz basis* of V_0 if there exist positive numbers c_1 and c_2 such that

$$c_1\left(\sum_{k\in\mathbb{Z}^n}|\alpha_k|^2\right)^{1/2}\le\left\|\sum_{k\in\mathbb{Z}^n}\alpha_k g_k\right\|_{L^2(\mathbb{R}^n)}\le c_2\left(\sum_{k\in\mathbb{Z}^n}|\alpha_k|^2\right)^{1/2} \tag{2.10}$$

for all ℓ^2-sequences (α_k).

The definition of a Riesz basis means that the mapping

$$(\alpha_k)\longmapsto\sum_{k\in\mathbb{Z}^n}\alpha_k g_k \tag{2.11}$$

defines a topological linear isomorphism from $\ell^2(\mathbb{Z}^n)$ onto V_0.

Remark 7. The anonymous referee kindly pointed out that if $\{g_\delta(x-k)\}_{\delta\in D,k\in\mathbb{Z}^n}$ is a Riesz basis of V_0, then the matrix

$$A(\xi):=\left(\sum_{k\in\mathbb{Z}^n}\widehat{g}_{\delta'}(\xi+2\pi k)\overline{\widehat{g}_{\delta''}(\xi+2\pi k)}\right)_{(\delta',\delta'')\in D\times D}, \tag{2.12}$$

is a hermitian invertible matrix satisfying

$$0<c_1Id\le A(\xi)\le c_2Id.$$

Hence one can consider $B(\xi):=(A(\xi))^{-1/2}$ and it can be checked that $\Phi(x)$ defined by

$$\widehat{\Phi}(\xi):=B(\xi)^t(\widehat{g}_\delta)_{\delta\in D}$$

satisfies condition (d) in Definition 5.

Definition 7. A multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ is said to be *r-regular* if the scaling function $\Phi(x)\in(V_0)^d$ appearing in part (d) of Definition 5 is r-regular.

Example 1. Let $\{\widetilde{V}_j\}_{j\in\mathbb{Z}}$ be a $(2r+1)$-regular multiresolution analysis in $L^2(\mathbb{R}^n)$ for single-wavelets. Then there exist a $(2r+1)$-regular scaling function φ and a $(2r+1)$-regular wavelet functions ψ_ε, $\varepsilon\in E$. Put $d=2^n$ and identify $D\simeq R$. Take $\Phi:={}^t(\psi_\varepsilon)_{\varepsilon\in R}$, where $\psi_0:=\varphi$, as an r-regular scaling function for multi-wavelets. Define $V_0:=\overline{\mathrm{Span}\{(\varphi_\delta)_{0k}\}}_{\delta\in D,k\in\mathbb{Z}^n}$ and V_j, $j\in\mathbb{Z}\backslash\{0\}$, by property (b) in Definition 5. Then $\{V_j\}_{j\in\mathbb{Z}}$ is an r-regular multiresolution analysis of $L^2(\mathbb{R}^n)$ for multi-wavelets.

Now we can state our main results. The first theorem deals with the case which is not necessarily r-regular.

Theorem 2. *Let a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of multi-wavelets be given. Then, there exists a family $\{\Psi_\varepsilon\}_{\varepsilon\in E}$ of 2^n-1 wavelet functions $\Psi_\varepsilon:={}^t(\psi_{\varepsilon1},\dots,\psi_{\varepsilon d})\in V_1^d$, $\varepsilon\in E$.*

The second theorem deals with the r-regular case.

Theorem 3. *Let an r-regular multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of multi-wavelets be given. Then, there exists an r-regular family $\{\Psi_\varepsilon\}_{\varepsilon\in E}$ of 2^n-1 wavelet functions $\Psi_\varepsilon:={}^t(\psi_{\varepsilon1},\dots,\psi_{\varepsilon d})\in V_1^d$, $\varepsilon\in E$.*

3. Proof of Theorem 1

Our proof of Theorem 1 is based on the following proposition.

Proposition 1. *Let X and Y be C^1-manifolds and $f : X \longrightarrow Y$ be a C^1-mapping. If X is compact and if $\dim X < \dim Y$, then the set $Y \backslash f(X)$ is open and dense in Y.*

We first prove Theorem 1 under the assumption that Proposition 1 is valid.

Proof of Theorem 1. We fix $n = \dim X$ and prove Theorem 1 by a double induction with respect to (m, d) satisfying the inequalities (2.1).

First step. (The case $d = 1$.) Let m be a positive integer satisfying $2m - n \geq 2d = 2$ and let $f_1 : X \longrightarrow \mathbb{C}^m$ be a C^∞-mapping with the property $|f_1(x)| = 1$ for each $x \in X$. Then we have a C^∞-mapping $f_1 : X \longrightarrow S^{2m-1} = \{z \in \mathbb{C}^m;\ |z| = 1\}$. Since $\dim X = n < 2m - 1 = \dim S^{2m-1}$, Proposition 1 implies that $S^{2m-1} \backslash f_1(X)$ is open and dence in S^{2m-1}. Thus we can find a point $y_0 \in S^{2m-1} \backslash f_1(X)$ and an open neighbourhood V_0 of y_0 in $\mathbb{C}^m$ such that $(V_0 \cap S^{2m-1}) \subset S^{2m-1} \backslash f_1(X)$. We choose a constant unitary matrix C such as $Ce_m = y_0$, and put $g_1 := C^{-1} \circ f_1$, $V_1 := C^{-1}(V_0)$. Then V_1 is an open neighbourhood of e_m in $\mathbb{C}^m$ such that

$$g_1(X) \subset S^{2m-1} \backslash V_1. \tag{3.1}$$

Now we apply Gröchenig's method (Gröchenig [8] or Meyer [15]). For $z = (z_j) \in \mathbb{C}^m$ and for a parameter $\alpha > 0$, we define the matrix $G_\alpha(z)$ by

$$G_\alpha(z) := \begin{pmatrix} z' & \alpha I_{m-1} \\ z_m & (z')^* \end{pmatrix}, \tag{3.2}$$

where $z' = {}^t(z_1, z_2, \ldots, z_{m-1})$. Since the cyclic permutation $\sigma = (1, m, m-1, \ldots, 2)$ has signature $(-1)^{m-1}$, we have

$$\begin{aligned} \det G_\alpha(z) =& (-1)^{m-1} \begin{vmatrix} \alpha I_{m-1} & z' \\ (z')^* & z_m \end{vmatrix} = (-1)^{m-1} \begin{vmatrix} \alpha I_{m-1} & z' \\ 0 & z_m - \alpha^{-1}|z'|^2 \end{vmatrix} \\ =& (-1)^{m-1} \alpha^{m-2} (\alpha z_m - |z'|^2), \end{aligned} \tag{3.3}$$

where $|z'|^2 = \sum_{j=1}^{m-1} |z_j|^2$. Then, we have the following claim:

Claim 1. *There exists a positive number α_0 such that $\det G_\alpha(z) \neq 0$ for each $\alpha \in]0, \alpha_0]$ and for each $z \in S^{2m-1} \backslash V_1$.*

Proof of Claim 1. For each α, put $N(\alpha) := \{z \in S^{2m-1};\ \det G_\alpha(z) = 0\}$. It suffices to show that there exists a positive number α_0 such that

$$\cup_{\alpha \in]0, \alpha_0]} N(\alpha) \subset V_1. \tag{3.4}$$

We give $\alpha > 0$ and $z \in N(\alpha)$. From (3.3), we have $\alpha z_m = |z'|^2 = 1 - |z_m|^2 \geq 0$, which implies that $z_m \in \mathbb{R}$ and $0 < z_m < 1$. Thus z_m is the positive root of the equation $\alpha t = 1 - t^2$, that is,

$$z_m = -\alpha/2 + \sqrt{1 + \alpha^2/4}. \tag{3.5}$$

Choose positive numbers ε and δ_0 such that

$$\{z \in \mathbb{C}^m;\ |z'| < \varepsilon,\ |z_m - 1| < \delta_0\} \subset V_1. \tag{3.6}$$

By the continuity of the function: $z_m \longmapsto \sqrt{1-|z_m|^2}$ on the interval $[0,1]$, we can find a positive number δ in $]0,\delta_0]$ such that $\sqrt{1-|z_m|^2} < \varepsilon$ if $0 \le 1 - z_m < \delta$. Since (3.5) implies

$$|z_m - 1| = 1 - z_m = 1 + \alpha/2 - \sqrt{1+\alpha^2/4} < \alpha/2,$$

choosing α_0 as $0 < \alpha_0 \le 2\delta$, we have that each $z \in \cup_{\alpha\in]0,\alpha_0]} N(\alpha)$ satisfies

$$|z_m - 1| < \alpha_0/2 \le \delta \quad \text{and} \quad |z'| = \sqrt{1-|z_m|^2} < \varepsilon.$$

Thus (3.6) yields $z \in V_1$. Therefore we get (3.4). This completes the proof of Claim 1. □

Let α_0 be the positive number as in Claim 1. Fix α in $]0,\alpha_0]$ and define C^∞-mappings $v_\ell : X \longrightarrow \mathbb{C}^m$, $\ell = 1,\dots,m$, by

$$\begin{aligned} v_1(x) &:= g_1(x) = \sum_{j=1}^m g_{j1}(x)\, e_j, \\ v_\ell(x) &:= \alpha e_{\ell-1} + \overline{g_{(\ell-1)1}(x)}\, e_m, \quad \ell = 2,\dots,m. \end{aligned} \tag{3.7}$$

Then, by the definition (3.2) of $G_\alpha(z)$, we have $(v_1(x),\dots,v_m(x)) = G_\alpha(g_1(x))$. So, by the inclusion (3.1) and by Claim 1, $(v_1(x),\dots,v_m(x))$ forms a basis of $\mathbb{C}^m$ for each x in X.

Now we apply the Gram-Schmidt orthonormalization process to $\{v_\ell\}_{\ell=1,\dots,m}$ and have the following lemma:

Lemma 1. *There exist C^∞-mappings $w_\ell : X \longrightarrow S^{2m-1}$, $\ell = 1,\dots,m$, with $w_1 = v_1 = g_1$ and with the following property* (3.8.k) *for all k in* $\{1,\dots,m\}$*:*

$$\begin{aligned} (w_\ell(x), w_{\ell'}(x)) &= \delta_{\ell\ell'}, \qquad \textit{for } \ell,\ell' \in \{1,\dots,k\},\ x \in X; \\ \mathrm{Span}\{w_1(x),\dots,w_k(x)\} &= \mathrm{Span}\{v_1(x),\dots,v_k(x)\}, \qquad x \in X. \end{aligned} \tag{3.8.k}$$

Proof of Lemma 1. We construct $w_1,\dots,w_m$ inductively as follows. First we put $w_1 := v_1$. Then w_1 is clearly a C^∞-mapping with property (3.8.1). Next let k be an integer with $2 \le k \le m$ and assume that there exist C^∞-mappings $w_1,\dots,w_{k-1} : X \longrightarrow S^{2m-1}$ with property (3.8.k-1). Put

$$\widetilde{w}_k(x) := v_k(x) - \sum_{j=1}^{k-1} a_j(x) w_j(x),$$

where $a_j : X \longrightarrow \mathbb{C}$, $j = 1,\dots,k-1$, are unknown functions to be determined. Since (3.8.k-1) implies

$$\begin{aligned} (\widetilde{w}_k(x), w_\ell(x)) &= (v_k(x), w_\ell(x)) - \sum_{j=1}^{k-1} a_j(x)(w_j(x), w_\ell(x)) \\ &= (v_k(x), w_\ell(x)) - a_\ell(x), \qquad \ell = 1,\dots,k-1, \end{aligned}$$

putting $a_\ell(x) := (v_k(x), w_\ell(x))$, $\ell = 1, \dots, k-1$, we get

$$(\widetilde{w}_k(x), w_\ell(x)) = 0 \qquad \text{for } \ell = 1, \dots, k-1;$$

$$a_j : X \longrightarrow \mathbb{C} \quad \text{is} \quad C^\infty \qquad \text{for } j = 1, \dots, k-1.$$

Thus $\widetilde{w}_k : X \longrightarrow \mathbb{C}^m$ is also C^∞. Since $v_1(x), \dots, v_k(x)$ are linearly independent over $\mathbb{C}$, it follows that

$$v_k(x) \notin \mathrm{Span}\{w_1(x), \dots, w_{k-1}(x)\} = \mathrm{Span}\{v_1(x), \dots, v_{k-1}(x)\},$$

which yields $\widetilde{w}_k(x) \neq 0$ for each x in X. Thus, if we put $w_k(x) := \widetilde{w}_k(x)/|\widetilde{w}_k(x)|$, then $w_k : X \longrightarrow S^{2m-1}$ is a C^∞-mapping. By this construction of w_k, it follows that the mappings $w_1, \dots, w_k$ satisfy the desired property (3.8.k). The proof of Lemma 1 is complete. □

Now we can finish the proof of Theorem 1 for the case $d = 1$. Let $w_1, \dots, w_m : X \longrightarrow S^{2m-1}$ be the mappings obtained in Lemma 1. Define the mappings $f_2, \dots, f_m : X \longrightarrow \mathbb{C}^m$ by

$$f_\ell(x) := C w_\ell(x), \qquad \ell = 2, \dots, m.$$

Then each f_ℓ is clearly C^∞ and we have

$$(f_1(x), \dots, f_m(x)) = C(w_1(x), \dots, w_m(x))$$

by $w_1(x) = v_1(x) = g_1(x) = C^{-1} f_1(x)$. Since C is unitary, we get

$$(f_k(x), f_\ell(x)) = (Cw_k(x), Cw_\ell(x)) = (w_k(x), w_\ell(x)) = \delta_{k\ell}, \quad k, \ell \in \{1, \dots, m\}.$$

This completes the proof of Theorem 1 for the case when $d = 1$.

Second step. (The case $d \geq 2$.) Let m and d be positive integers satisfying

$$4 \leq 2d \leq 2m - n, \tag{3.9}$$

and let $f_\ell : X \longrightarrow \mathbb{C}^m$, $\ell = 1, \dots, d$, be C^∞-mappings with the property

$$(f_k(x), f_\ell(x)) = \delta_{k\ell}, \qquad \text{for } k, \ell \in \{1, \dots, d\},\ x \in X. \tag{3.10}$$

Put $g_\ell := f_\ell$, $\ell = 1, \dots, d-1$. Since (3.9) implies $2 \leq 2(d-1) \leq 2m - n$, Theorem 1 is valid for $(m, d-1)$ by the inductive assumption. Thus, there exist C^∞-mappings $g_\ell : X \longrightarrow \mathbb{C}^m$, $\ell = d, \dots, m$, with the property

$$(g_k(x), g_\ell(x)) = \delta_{k\ell}, \qquad \text{for } k, \ell \in \{1, \dots, m\},\ x \in X. \tag{3.11}$$

Since (3.11) means that $\{g_\ell\}_{\ell=1,\dots,m}$ forms an orthonormal basis of $\mathbb{C}^m$ for each $x \in X$, there exist uniquely determined mappings $h_{jd} : X \longrightarrow \mathbb{C}$, $j = 1, \dots, m$ such that

$$f_d(x) = \sum_{j=1}^m h_{jd}(x) g_j(x). \tag{3.12}$$

Since (3.11) and (3.12) imply

$$h_{jd}(x) = \sum_{k=1}^{m} h_{kd}(x)(g_k(x), g_j(x)) = (f_d(x), g_j(x)),$$

$h_{jd} : X \longrightarrow \mathbb{C}$ is C^∞, for $j = 1, \ldots, m$, and $h_{jd} \equiv 0$ for $j = 1, \ldots, d-1$ by (3.10). Then (3.12) can be written as

$$f_d(x) = \sum_{j=d}^{m} h_{jd}(x) g_j(x). \tag{3.12'}$$

Note that (3.10), (3.11), and (3.12') also yield that

$$\begin{aligned}\sum_{j=d}^{m} |h_{jd}(x)|^2 =& \sum_{j=d}^{m}\sum_{k=d}^{m} h_{jd}(x)\overline{h_{kd}(x)}(g_j(x), g_k(x)) \\ =&(f_d(x), f_d(x)) = 1.\end{aligned}$$

Therefore we get a C^∞-mapping $h_d = (h_{jd}; j \downarrow d, \ldots, m) : X \longrightarrow S^{2(m-d+1)-1}$. Since (3.9) implies $2 \leq 2(m-d+1) - n$, Theorem 1 is also valid for $(m-d+1, 1)$ by the inductive assumption. Thus, there exist C^∞-mappings $h_\ell : X \longrightarrow \mathbb{C}^{m-d+1}$, $\ell = d+1, \ldots, m$, with the property

$$(h_k(x), h_\ell(x))_{\mathbb{C}^{m-d+1}} = \delta_{k\ell}, \qquad k, \ell \in \{d, \ldots, m\},\ x \in X. \tag{3.13}$$

Put $h(x) = (h_d(x), \ldots, h_m(x))$, where $h_\ell(x) = \sum_{j=d}^{m} h_{j\ell}(x) e_j$ for $\ell \in \{d, \ldots, m\}$. Then, for each $x \in X$, we get

$$\begin{aligned}U(m) \ni&(g_1(x), \ldots, g_m(x)) \begin{pmatrix} I_{d-1} & 0 \\ 0 & h(x) \end{pmatrix} \\ =&(g_1(x), \ldots, g_{d-1}(x), \sum_{j=d}^{m} h_{jd}(x) g_j(x), \ldots, \sum_{j=d}^{m} h_{jm}(x) g_j(x)) \\ =&(f_1(x), \ldots, f_d(x), \sum_{j=d}^{m} h_{j(d+1)}(x) g_j(x), \ldots, \sum_{j=d}^{m} h_{jm}(x) g_j(x)).\end{aligned}$$

Thus, if we define mappings $f_\ell : X \longrightarrow \mathbb{C}^m$, $\ell = d+1, \ldots, m$ by $f_\ell(x) := \sum_{j=d}^{m} h_{j\ell}(x) g_j(x)$, then the mappings $f_{d+1}, \ldots, f_m$ are C^∞ with the desired property

$$(f_k(x), f_\ell(x)) = \delta_{k\ell}, \qquad k, \ell \in \{1, \ldots, m\},\ x \in X.$$

This completes the proof of Theorem 1. □

Now we shall prove Proposition 1. Let X and Y be C^1-manifolds and let $f : X \longrightarrow Y$ be a C^1-mapping. We assume that X is compact and that $\dim X < \dim Y$. Then $f(X)$ is also compact; so it is closed in Y. Thus, the complement $Y \backslash f(X)$ is open in Y. Therefore, it suffices for Proposition 1 to show the following proposition.

Proposition 2. *Let X and Y be C^1-manifolds and let $f : X \longrightarrow Y$ be a C^1-mapping. If $\dim X < \dim Y$, then $Y \backslash f(X)$ is dense in Y.*

To show Proposition 2, we first recall the following lemma.

Lemma 2. *(Lindelöf) Let X be a topological space satisfying the second axiom of countability. Then X has the Lindelöf property, that is, each open covering of X has a countable subcovering.*

We prove Proposition 2 by using the notion of *measure zero*. To define this notion on manifolds we need the following well-known lemma in differential geometry. See Sternberg [16, Chapter 2, Section 3, (3.2)].

Lemma 3. *Let U be an open set in $\mathbb{R}^p$ and let $\varphi : U \longrightarrow \mathbb{R}^p$ be a C^1-mapping. If a subset A of U is of measure zero, then $\varphi(A)$ is also of measure zero in $\mathbb{R}^p$.*

By virtue of Lemma 3, the following definition makes sense.

Definition 8. Let Y be a p-dimensional C^1-manifold. A subset B of Y is said to be *of measure zero* if there exists an atlas $\{(V_j, \psi_j)\}_{j\in\mathbb{N}}$ of Y such that each $\psi_j(V_j \cap B)$ is of measure zero in $\mathbb{R}^p$.

Remark 8. Definition 8 is independent of the choice of the atlas $\{(V_j, \psi_j)\}_{j\in\mathbb{N}}$ of Y, which is a direct consequence of Lemma 3.

Remark 9. If a subset B of a manifold Y is of measure zero, then $Y \backslash B$ is dense in Y.

Proof. Assume the assertion were false. Then there would exist an open set W in Y such that $W \subset B$. We take a point $y \in W$ and a local chart (V, ψ) satisfying $y \in V$. Then $\emptyset \neq (V \cap W) \subset (V \cap B)$, so that

$$\emptyset \neq \psi(V \cap W) \subset \psi(V \cap B). \tag{3.14}$$

Since $\psi(V \cap W)$ is a non-empty open set in $\mathbb{R}^p$, $p = \dim Y$, it has a positive measure. Then (3.14) implies that $\psi(V \cap B)$ also has a positive measure, which contradicts the fact that $\psi(V \cap B)$ is of measure zero. □

By virtue of Remark 9, to prove Proposition 2 it suffices to show the following proposition.

Proposition 3. *Let X and Y be C^1-manifolds and let $f : X \longrightarrow Y$ be a C^1-mapping. If $n = \dim X < \dim Y = p$, then $f(X)$ is of measure zero in Y.*

Proof. By Definition 8, it suffices to show that, for every local chart (V, ψ) of Y, $\psi(f(X) \cap V)$ is of measure zero in $\mathbb{R}^p$. For each $x \in f^{-1}(V)$, choosing a local chart (U_x, φ_x) of X such that $x \in U_x$ and $f(U_x) \subset V$, we have an open covering $\{U_x\}_{x\in f^{-1}(V)}$ of $f^{-1}(V)$. Since $f^{-1}(V)$ satisfies the second axiom of countability, Lemma 2 implies that there exists a countable open subcovering $\{U_{x_j}\}_{j\in\mathbb{N}}$. Then we have

$$V \cap f(X) = f(f^{-1}(V)) = f(\cup_{j\in\mathbb{N}} U_{x_j}) = \cup_{j\in\mathbb{N}} f(U_{x_j}). \tag{3.15}$$

For each $j \in \mathbb{N}$, we consider the following commutative diagram:

$$\begin{array}{ccc} X \supset U_{x_j} & \xrightarrow{\quad f \quad} & f(U_{x_j}) \subset V \subset Y \\ {\scriptstyle \varphi_{x_j}} \Big\downarrow & & \Big\downarrow {\scriptstyle \psi} \\ \mathbb{R}^n \supset \varphi_{x_j}(U_{x_j}) & \xrightarrow[f_j = \psi f \varphi_{x_j}^{-1}]{} & \psi f(U_{x_j}) \subset \psi(V) \subset \mathbb{R}^p. \end{array} \tag{3.16}$$

We put $\widetilde{U}_j := \varphi_{x_j}(U_{x_j}) \times \mathbb{R}^{p-n}$ and define a C^1-mapping $g_j : \widetilde{U}_j \longrightarrow \mathbb{R}^p$ by $g_j(x,y) := f_j(x)$ for $(x,y) \in \varphi_{x_j}(U_{x_j}) \times \mathbb{R}^{p-n}$. Since $\varphi_{x_j}(U_{x_j}) \times \{0\}$ is of measure zero in $\widetilde{U}_j$ by $n < p$, Lemma 3 implies that $g_j(\varphi_{x_j}(U_{x_j}) \times \{0\})$ is of measure zero in $\mathbb{R}^p$. Then, by the equality $g_j(\varphi_{x_j}(U_{x_j}) \times \{0\}) = f_j(\varphi_{x_j}(U_{x_j})) = \psi f(U_{x_j})$, we have that each $\psi f(U_{x_j})$ is of measure zero in $\mathbb{R}^p$. Since (3.15) implies $\psi(V \cap f(X)) = \cup_{j\in\mathbb{N}} \psi f(U_{x_j})$, we conclude that $\psi(V \cap f(X))$ is also of measure zero in $\mathbb{R}^p$. This completes the proof of Proposition 3. □

4. Basic Properties of Multi-wavelets

Hereafter we assume that we have a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ with a scaling function Φ as given in Definition 5, (d).

Notation 3. Given a function $F(x) := (f_1(x), \dots, f_d(x)) \in L^2(\mathbb{R}^n_x)^d$, denote its Fourier transform by $\widehat{F}(\xi) := (\widehat{f}_1(\xi), \dots, \widehat{f}_d(\xi)) \in L^2(\mathbb{R}^n_\xi)^d$.

Lemma 4. *There exists an $L^2(\mathbb{T}^n)$-valued matrix*

$$\begin{aligned} M_0(\xi) :=& \big((m_0)_{(d',d'')}(\xi); d' \downarrow 1, \dots, d, d'' \to 1, \dots, d\big) \\ =& \big((m_0)_{(d',d'')}(\xi)\big)_{(d',d'')\in D\times D} \in Mat(d \times d; L^2(\mathbb{T}^n)) \end{aligned}$$

such that

$$\widehat{\Phi}(2\xi) = M_0(\xi)\widehat{\Phi}(\xi). \tag{4.1}$$

Proof. Since we have the inclusions $\varphi_{d''}\left(2^{-1}x\right) \in V_{-1} \subset V_0$, $d'' \in D$, then $\varphi_{d''}\left(2^{-1}x\right)$ can be expanded in terms of the basis $\{\varphi_{d'}(x-k)\}_{d'\in D, k\in\mathbb{Z}^n}$ of V_0,

$$\varphi_{d''}\left(2^{-1}x\right) = \sum_{d'\in D, k\in\mathbb{Z}^n} \beta_{d''d'k}\varphi_{d'}(x-k), \tag{4.2}$$

where the coefficients $\beta_{d''d'k}$ are defined by the scalar product

$$\beta_{d''d'k} := \left(\varphi_{d''}(2^{-1}x), \varphi_{d'}(x-k)\right)_{L^2(\mathbb{R}^n)} \tag{4.3}$$

and the sequence $(\beta_{d''d'k})_{k\in\mathbb{Z}^n}$ belongs to $\ell^2(\mathbb{Z}^n)$. Taking the Fourier transform of (4.2) we have

$$2^n\widehat{\varphi}_{d''}(2\xi) = \sum_{d'\in D, k\in\mathbb{Z}^n} \beta_{d''d'k}\widehat{\varphi}_{d'}(\xi)\, e^{-ik\cdot\xi}. \tag{4.4}$$

If we put

$$(m_0)_{(d',d'')}(\xi) := 2^{-n} \sum_{k\in\mathbb{Z}^n} \beta_{d''d'k}\, e^{-ik\cdot\xi}, \tag{4.5}$$

then $(m_0)_{(d',d'')}(\xi)$ is in $L^2(\mathbb{T}^n)$ and satisfies (4.1). □

Notation 4. Let $d\mu(\xi) := (2\pi)^{-n}\, d\xi$ denote the normalized Haar measure of the torus $\mathbb{T}^n$ and I_d denote the identity matrix of order d.

Lemma 5. *The sequence* $\{\varphi_\delta(x-k)\}_{\delta\in D,k\in\mathbb{Z}^n}$ *is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}^n}\widehat{\Phi}(\xi+2\pi k)^t\overline{\widehat{\Phi}(\xi+2\pi k)}\equiv I_d. \qquad \text{a.a. } \xi, \tag{4.6}$$

Proof. Put

$$G_{(d',d'')}(\xi):=\sum_{k\in\mathbb{Z}^n}\widehat{\varphi}_{d'}(\xi+2\pi k)\overline{\widehat{\varphi}_{d''}(\xi+2\pi k)}.$$

Since $G_{(d',d'')}(\xi)\in L^1(\mathbb{T}^n)$, then its Fourier series, in the sense of distributions in $\mathcal{D}'(\mathbb{T}^n)$, is

$$G_{(d',d'')}(\xi)=\sum_{l\in\mathbb{Z}^n}\widehat{G}_{(d',d'')}(l)\,e^{il\cdot\xi},$$

where

$$\widehat{G}_{(d',d'')}(l):=\int_{\mathbb{T}^n}e^{-il\cdot\xi}G_{(d',d'')}(\xi)\,d\mu(\xi).$$

On the other hand, the orthonormality of $\{\varphi_\delta(x-k)\}_{\delta\in D,k\in\mathbb{Z}^n}$ is equivalent to

$$\begin{aligned}\delta_{l,0}\delta_{d',d''}=&\int_{\mathbb{R}^n}\varphi_{d'}(x-l)\overline{\varphi_{d''}(x)}\,dx\\ =&(2\pi)^{-n}\int_{\mathbb{R}^n}e^{-il\cdot\xi}\widehat{\varphi}_{d'}(\xi)\overline{\widehat{\varphi}_{d''}(\xi)}\,d\xi\\ =&\sum_{k\in\mathbb{Z}^n}(2\pi)^{-n}\int_{[0,2\pi]^n}e^{-il\cdot(\xi+2\pi k)}\widehat{\varphi}_{d'}(\xi+2\pi k)\overline{\widehat{\varphi}_{d''}(\xi+2\pi k)}\,d\xi\\ =&\int_{\mathbb{T}^n}e^{-il\cdot\xi}G_{(d',d'')}(\xi)\,d\mu(\xi)=\widehat{G}_{(d',d'')}(l), \qquad l\in\mathbb{Z}^n,\end{aligned}$$

which, in turn, is equivalent to

$$G_{(d',d'')}(\xi)=\sum_{l\in\mathbb{Z}^n}\delta_{l,0}\delta_{d',d''}\,e^{il\cdot\xi}\equiv\delta_{d',d''}. \quad \square$$

Lemma 6. *The matrix* $M_0(\xi)$*, defined by* (4.1)*, i.e.* $\widehat{\Phi}(2\xi)=M_0(\xi)\widehat{\Phi}(\xi)$*, satisfies*

$$\sum_{\eta\in R}M_0(\xi+\pi\eta)M_0(\xi+\pi\eta)^*\equiv I_d, \qquad \text{a.a. } \xi, \tag{4.7}$$

where M_0^* *denotes the adjoint of* M_0.

Proof. By Lemma 5 we have

$$\sum_{k\in\mathbb{Z}^n}\widehat{\Phi}(2\xi+2\pi k)\widehat{\Phi}(2\xi+2\pi k)^*\equiv I_d, \qquad \text{a.a. } \xi,$$

and, by Lemma 4,

$$\widehat{\Phi}(2\xi+2\pi k)=M_0(\xi+\pi k)\widehat{\Phi}(\xi+\pi k).$$

Put $k = 2l + \eta$ with $k,\, l \in \mathbb{Z}^n$ and $\eta \in R$. Since $\mathbb{Z}^n = 2\mathbb{Z}^n + R$ and M_0 is $2\pi\mathbb{Z}^n$-periodic, then

$$\begin{aligned}
I_d &\equiv \sum_{k\in\mathbb{Z}^n} M_0(\xi+\pi k)\widehat{\Phi}(\xi+\pi k)(M_0(\xi+\pi k)\widehat{\Phi}(\xi+\pi k))^* \\
&= \sum_{l\in\mathbb{Z}^n,\eta\in R} M_0(\xi+2\pi l+\pi\eta)\widehat{\Phi}(\xi+2\pi l+\pi\eta) \\
&\quad \times \widehat{\Phi}(\xi+2\pi l+\pi\eta)^* M_0(\xi+2\pi l+\pi\eta)^* \\
&= \sum_{\eta\in R} M_0(\xi+\pi\eta)\Bigg(\sum_{l\in\mathbb{Z}^n}\widehat{\Phi}((\xi+\pi\eta)+2\pi l)\widehat{\Phi}((\xi+\pi\eta)+2\pi l)^*\Bigg)M_0(\xi+\pi\eta)^* \\
&= \sum_{\eta\in R} M_0(\xi+\pi\eta)M_0(\xi+\pi\eta)^*,
\end{aligned}$$

because $\sum_{l\in\mathbb{Z}^n}\widehat{\Phi}((\xi+\pi\eta)+2\pi l)\widehat{\Phi}((\xi+\pi\eta)+2\pi l)^* \equiv I_d$ by Lemma 5. □

Let $\{\varphi_\delta(x-k)\}_{\delta\in D,k\in\mathbb{Z}^n}$ be an orthonormal basis of V_0. Then there exists a natural isomorphism ι between the Hilbert spaces V_0 and $\ell^2(\mathbb{Z}^n)^d$:

$$\iota : V_0 \ni f \longmapsto \big((\alpha_{f\delta k})_{k\in\mathbb{Z}^n}\big)_{\delta\in D} \in \ell^2(\mathbb{Z}^n)^d, \tag{4.8}$$

defined by the formula:

$$f(x) = \sum_{\delta\in D,k\in\mathbb{Z}^n} \alpha_{f\delta k}\varphi_\delta(x-k), \qquad \alpha_{f\delta k} := (f(x),\varphi_\delta(x-k))_{L^2(\mathbb{R}^n)}. \tag{4.9}$$

Notation 5. For $f \in V_0$ and $\iota(f) = ((\alpha_{f\delta k})_{k\in\mathbb{Z}^n})_{\delta\in D}$ given by (4.8) and (4.9), set

$$m(f) := \big(m(f)_\delta\big)_{\delta\in D}, \quad \text{where} \quad m(f)_\delta(\xi) := \sum_{k\in\mathbb{Z}^n} \alpha_{f\delta k}\, e^{-ik\cdot\xi}, \quad \delta\in D. \tag{4.10}$$

According to notation (4.10), we have

$$\widehat{f}(\xi) = m(f)(\xi)\widehat{\Phi}(\xi)$$

and

$$M_0(\xi) = \big(2^{-n} m(\varphi_{d''}(x/2))_{d'}(\xi)\big)_{(d',d'')\in D\times D}.$$

Denote by $\mathcal{F}$ the Fourier transformation in $L^2(\mathbb{R}^n)$ and by

$$L^2(\mathbb{T}^n)^d := (L^2(\mathbb{T}^n),\ldots,L^2(\mathbb{T}^n))$$

the d-fold product Hilbert space of $L^2(\mathbb{T}^n)$ with the inner product

$$(\cdot,\cdot)_{L^2(\mathbb{T}^n)^d} := (\cdot,\cdot)_{L^2(\mathbb{T}^n)} + \cdots + (\cdot,\cdot)_{L^2(\mathbb{T}^n)}.$$

Lemma 7. *The Fourier transforms of* V_0 *and* V_{-1} *satisfy the relations, respectively,*

$$\mathcal{F}V_0 = L^2(\mathbb{T}^n)^d\widehat{\Phi}(\xi), \qquad \mathcal{F}V_{-1} = L^2(2\mathbb{T}^n)^d M_0(\xi)\widehat{\Phi}(\xi). \tag{4.11}$$

Proof. Since the Fourier transformation is a constant multiple of a unitary operator, then $\widehat{f}(\xi) = m(f)(\xi)\widehat{\Phi}(\xi)$ defines a natural isomorphism between the Hilbert spaces $\mathcal{F}V_0$ and $L^2(\mathbb{T}^n)^d$:

$$\mathcal{F}V_0 \ni \widehat{f} \longmapsto \big(m(f)_\delta\big)_{\delta\in D} \in L^2(\mathbb{T}^n)^d.$$

By Part (b) of Definition 5, $\widehat{f}(\xi) \in \mathcal{F}V_0$ if and only if $\widehat{f}(2\xi) \in \mathcal{F}V_{-1}$. Hence, by Lemma 4,

$$\widehat{f}(2\xi) = m(f)(2\xi)\widehat{\Phi}(2\xi) = m(f)(2\xi)M_0(\xi)\widehat{\Phi}(\xi). \quad \square$$

Lemma 8. *Let* $f, g \in V_0$. *Then*

$$(f,g)_{L^2(\mathbb{R}^n;dx)} = (m(f)\widehat{\Phi}, m(g)\widehat{\Phi})_{L^2(\mathbb{R}^n;d\mu(\xi))} = (m(f), m(g))_{L^2(\mathbb{T}^n;d\mu(\xi))^d}. \tag{4.12}$$

Proof. Since every element of $m(f)$ and $m(g)$ is $2\pi\mathbb{Z}^n$-periodic, Lemma 5 implies that

$$\begin{aligned}
(f,g)_{L^2(\mathbb{R}^n;dx)} &= (\widehat{f},\widehat{g})_{L^2(\mathbb{R}^n;d\mu(\xi))} \\
&= \int_{\mathbb{R}^n} m(f)(\xi)\widehat{\Phi}(\xi)\overline{m(g)(\xi)\widehat{\Phi}(\xi)}\, d\mu(\xi) \\
&= \int_{\mathbb{R}^n} m(f)(\xi)\widehat{\Phi}(\xi)^t\overline{\widehat{\Phi}(\xi)}^t\overline{m(g)(\xi)}\, d\mu(\xi) \\
&= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n} m(f)(\xi)\widehat{\Phi}(\xi+2\pi k)^t\overline{\widehat{\Phi}(\xi+2\pi k)}^t\overline{m(g)(\xi)}\, d\mu(\xi) \\
&= (m(f), m(g))_{L^2(\mathbb{T}^n;d\mu(\xi))^d}. \quad \square
\end{aligned}$$

Remark 10. By Lemma 7 and Lemma 8, we have

$$\mathcal{F}V_0 \simeq L^2(\mathbb{T}^n)^d \quad \text{and} \quad \mathcal{F}V_{-1} \simeq L^2(2\mathbb{T}^n)^d M_0(\xi).$$

Notation 10. Denote the orthogonal complement of $L^2(2\mathbb{T}^n)^d M_0(\xi)$ in $L^2(\mathbb{T}^n)^d$ by

$$\widetilde{W}_{-1} := (L^2(2\mathbb{T}^n)^d M_0(\xi))^\perp \qquad \text{in } L^2(\mathbb{T}^n)^d. \tag{4.13}$$

A family of wavelet functions is an orthonormal system for the orthogonal complement, $(V_{-1})^\perp$, of V_{-1} in V_0; this subspace is isomorphic to the orthogonal complement, $(\mathcal{F}V_{-1})^\perp$, of $\mathcal{F}V_{-1}$ in $\mathcal{F}V_0$. Moreover, $(\mathcal{F}V_{-1})^\perp$ is isomorphic to $\widetilde{W}_{-1}$ by Lemma 8.

Lemma 9. *The orthogonal complements* $\widetilde{W}_{-1}$ *and* $(\mathcal{F}V_{-1})^\perp$ *satisfy the following relations:*

$$\widetilde{W}_{-1} = \Big\{l(\xi) \in L^2(\mathbb{T}^n)^d; \sum_{\eta\in R} M_0(\xi+\pi\eta)l(\xi+\pi\eta)^* \equiv 0 \text{ a.a. } \xi\Big\}, \tag{4.14}$$

$$\widetilde{W}_{-1}\widehat{\Phi}(\xi) = (\mathcal{F}V_{-1})^\perp \quad (\text{in } \ \mathcal{F}V_0). \tag{4.15}$$

Proof. Let $l(\xi) \in \widetilde{W}_{-1}$. Then for every $n(\xi) \in L^2(\mathbb{T}^n)^d$, that is, $n(2\xi) \in L^2(2\mathbb{T}^n)^d$,

$$\big(n(2\xi)M_0(\xi), l(\xi)\big)_{L^2(\mathbb{T}^n)^d} = 0.$$

Put $\xi = \xi' + \pi\eta$, $\xi \in \mathbb{T}^n$, $\xi' \in 2\mathbb{T}^n$, and $\eta \in R$. Since $\mathbb{T}^n = 2\mathbb{T}^n + \pi R$, then

$$\begin{aligned} 0 &= \int_{\mathbb{T}^n} n(2\xi)M_0(\xi)l(\xi)^*\, d\mu(\xi) \\ &= \sum_{\eta\in R}\int_{2\mathbb{T}^n} n(2\xi' + 2\pi\eta)M_0(\xi'+\pi\eta)l(\xi'+\pi\eta)^*\, d\mu(\xi') \\ &= \int_{2\mathbb{T}^n} n(2\xi')\sum_{\eta\in R} M_0(\xi'+\pi\eta)l(\xi'+\pi\eta)^*\, d\mu(\xi'). \end{aligned}$$

Since $\sum_{\eta\in R} M_0(\xi'+\pi\eta)l(\xi'+\pi\eta)^*$ is $\pi\mathbb{Z}^n$-periodic and $n(2\xi)$ is an arbitrary function in $L^2(2\mathbb{T}^n)^d$, then

$$\sum_{\eta\in R} M_0(\xi'+\pi\eta)l(\xi'+\pi\eta)^* \equiv 0$$

in $L^2(2\mathbb{T}^n)^d$, that is, for almost all ξ. □

Remark 11. The number of wavelet functions is $2^n - 1$ because relation (4.14) for $\widetilde{W}_{-1}$ defines a hyperplane of $\mathbb{C}^d$-coefficients in $(\mathbb{C}^d)^{2^n}$:

$$\Big\{(z_\eta)_{\eta\in R} \in (\mathbb{C}^d)^{2^n};\ \sum_{\eta\in R} M_0(\xi+\pi\eta)z_\eta = 0\Big\} \tag{4.16}$$

for almost every fixed $\xi \in \mathbb{R}^n$, and there exist $d(2^n-1)$ orthonormal vectors in this hyperplane embedded in the vector space $\mathbb{C}^{2^n d}$.

Notation 7. Let $J_n = \{0, 1, \dots, 2^n - 1\}$. Any number $\ell \in J_n$ can be written uniquely, in the base two, as

$$\ell = c_{n-1}(\ell)2^{n-1} + c_{n-2}(\ell)2^{n-2} + \dots + c_1(\ell)2^1 + c_0(\ell), \tag{4.17}$$

where each $c_k(\ell)$, $k = 0, \dots, n-1$, is either 0 or 1. Write

$$\alpha_{n,\ell} = \big(c_{n-1}(\ell), c_{n-2}(\ell), \dots, c_1(\ell), c_0(\ell)\big), \qquad \ell \in J_n. \tag{4.18}$$

Hereafter we let $\{\alpha_{n,\ell}\}_{\ell\in J_n}$ define the ordering of R; we write $\Psi_0 := \Phi$ and, for short, $\Psi_{\alpha_{n,\ell}} = \Psi_\ell = \big(\psi_{\ell\delta}\big)_{\delta\in D} \in L^2(\mathbb{T}^n)^d$ for $\ell \in J_n \setminus \{0\}$.

Notation 8. For $M_\ell(\xi) := \big((m_\ell)_{(d',d'')}(\xi)\big)_{(d',d'')\in D\times D} \in \operatorname{Mat}(d\times d; L^2(\mathbb{T}^n))$, $\ell \in J_n$, satisfying $\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi)\widehat{\Phi}(\xi)$, let

$$L_{\ell'\ell''}(2\xi) := \sum_{\eta\in R} M_{\ell'}(\xi+\pi\eta)M_{\ell''}(\xi+\pi\eta)^*, \qquad \ell', \ell'' \in J_n \tag{4.19}$$

and

$$L(\xi) := \big(L_{\ell'\ell''}(\xi)\big)_{(\ell',\ell'')\in J_n\times J_n}. \tag{4.20}$$

We see that, for every $\eta' \in R$,

$$L_{\ell'\ell''}(2(\xi+\pi\eta')) = \sum_{\eta\in R} M_{\ell'}(\xi+\pi\eta'+\pi\eta)M_{\ell''}(\xi+\pi\eta'+\pi\eta)^* = L_{\ell'\ell''}(2\xi).$$

Hence, $L_{\ell'\ell''}(2\xi)$ is $\pi\mathbb{Z}^n$-periodic, and this implies that

$$L_{\ell'\ell''}(\xi) \in \mathrm{Mat}(d\times d; L^1(\mathbb{T}^n)).$$

and

$$L(\xi) \in \mathrm{Mat}\big(2^n\times 2^n; \mathrm{Mat}(d\times d; L^1(\mathbb{T}^n))\big) \simeq \mathrm{Mat}(2^n d\times 2^n d; L^1(\mathbb{T}^n)).$$

Lemma 10. *The sequence* $\{(\psi_{\ell\delta})_{0k}\}_{\ell\in J_n,\delta\in D,k\in\mathbb{Z}^n}$ *is an orthonormal system if and only if*

$$L(\xi) = I_{2^n d} \qquad \text{a.a. } \xi. \tag{4.21}$$

Proof. The sequence $\{(\psi_{\ell\delta})_{0k}\}_{\ell\in J_n,\delta\in D,k\in\mathbb{Z}^n}$ is an orthonormal system if and only if, for every $k\in\mathbb{Z}^n$ and for $\ell',\ell''\in J_n$,

$$\begin{aligned}\delta_{(\ell',k),(\ell'',0)} I_d &= \int_{\mathbb{R}^n} (\Psi_{\ell'})_{0k} (\Psi_{\ell''})_{00}^* \, dx \\ &= \int_{\mathbb{R}^n} \widehat{(\Psi_{\ell'})_{0k}}\, \widehat{(\Psi_{\ell''})_{00}}^{\,*} \, d\mu(\xi).\end{aligned}$$

If we put $\xi = 2\zeta$ and substitute

$$\widehat{\Psi}_\ell(2\zeta) = M_\ell(\zeta)\widehat{\Phi}(\zeta), \qquad \ell = \ell', \ell'',$$

in the above, then

$$\begin{aligned}\delta_{(\ell',k),(\ell'',0)} I_d &= \int_{\mathbb{R}^n} e^{-2ik\cdot\zeta} M_{\ell'}(\zeta)\widehat{\Phi}(\zeta)\widehat{\Phi}(\zeta)^* M_{\ell''}(\zeta)^* \, d\mu(2\zeta) \\ &= \sum_{k'\in\mathbb{Z}^n} \int_{2\mathbb{T}^n} e^{-2ik\cdot\zeta} M_{\ell'}(\zeta+\pi k')\widehat{\Phi}(\zeta+\pi k')\widehat{\Phi}(\zeta+\pi k')^* \\ &\qquad\qquad \times M_{\ell''}(\zeta+\pi k')^* \, d\mu(2\zeta).\end{aligned}$$

Now, put $k' = 2l+\eta$ with k', $l\in\mathbb{Z}^n$ and $\eta\in R$. Since $\mathbb{Z}^n = 2\mathbb{Z}^n + R$, and M_ℓ, $\ell\in J_n$, are $2\pi\mathbb{Z}^n$-periodic, then

$$\begin{aligned}\delta_{(\ell',k),(\ell'',0)} I_d &= \sum_{l\in\mathbb{Z}^n,\eta\in R} \int_{2\mathbb{T}^n} e^{-2ik\cdot\zeta} M_{\ell'}\big(\zeta+\pi(2l+\eta)\big) \\ &\qquad \times \widehat{\Phi}(\zeta+\pi(2l+\eta))\widehat{\Phi}(\zeta+\pi(2l+\eta))^* M_{\ell''}\big(\zeta+\pi(2l+\eta)\big)^* \, d\mu(2\zeta) \\ &= \int_{2\mathbb{T}^n} e^{-2ik\cdot\zeta} \sum_{\eta\in R} M_{\ell'}(\zeta+\pi\eta) \\ &\qquad \times \sum_{l\in\mathbb{Z}^n} \widehat{\Phi}\big((\zeta+2l)+\pi\eta\big)\widehat{\Phi}\big((\zeta+2l)+\pi\eta\big)^* M_{\ell''}(\zeta+\pi\eta), d\mu(2\zeta) \\ &= \int_{2\mathbb{T}^n} e^{-2ik\cdot\zeta} L_{\ell'\ell''}(2\zeta)\, d\mu(2\zeta) \\ &= \int_{\mathbb{T}^n} e^{-ik\cdot\xi} L_{\ell'\ell''}(\xi)\, d\mu(\xi) = \widehat{L}_{\ell'\ell''}(k), \qquad k\in\mathbb{Z}^n.\end{aligned}$$

Hence the sequence $\{(\psi_{\ell\delta})_{0k}\}_{\ell\in J_n,\delta\in D,k\in\mathbb{Z}^n}$ is an orthonormal system if and only if

$$L_{\ell'\ell''}(\xi) \equiv \delta_{\ell'\ell''} I_d, \quad \text{a.a. } \xi, \qquad \ell',\, \ell''\in J_n. \quad \square$$

Notation 9. Let $M_\ell(\xi)$ be the same as in Notation 8. Put

$$\widetilde{M}_{\ell' d'}(\xi) := \big((m_{\ell'})_{(d'',d')}(\xi+\pi\alpha_{n,\ell''})\big)_{(d'',\ell'')\in D\times J_n} \in \mathrm{Mat}(d\times 2^n; L^2(\mathbb{T}^n)), \tag{4.22}$$

for $\ell' \in J_n$, $d' \in D$ and put

$$\widetilde{M}(\xi) := \big(\widetilde{M}_{\ell' d'}(\xi)\big)_{(\ell',d')\in J_n\times D} \in \mathrm{Mat}\big(2^n\times d; \mathrm{Mat}(d\times 2^n; L^2(\mathbb{T}^n))\big). \tag{4.23}$$

Remark 12. The matrix $\widetilde{M}(\xi)$ is given by changing the order of the columns of the matrix $\big(M_{\ell'}(\xi+\pi\alpha_{n,\ell''})\big)_{(\ell',\ell'')\in J_n\times J_n}$ from the ordered set $J_n\times D$ with lexicographic order to the ordered set $D\times J_n$ with lexicographic order.

Then we have

$$\widetilde{M}(\xi)\widetilde{M}(\xi)^* = L(2\xi). \tag{4.24}$$

Proposition 4. *The family of functions $\{\Psi_\ell\}_{\ell\in J_n\setminus\{0\}}$ defined by the relations $\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi)\widehat{\Phi}(\xi)$, is a family of wavelet functions if and only if*

$$\widetilde{M}(\xi) \in U(2^n d), \quad \text{a.a. } \xi. \tag{4.25}$$

Proof. By (4.24), (4.25) is equivalent to (4.21). Since we assume the existence of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$, then $\{\Psi_\ell\}_{\ell\in J_n\setminus\{0\}}$ is a family of wavelet functions if and only if $\{(\psi_{\ell\delta})_{jk}\}_{\ell\in J_n,\delta\in D,k\in\mathbb{Z}^n}$ is an orthonormal basis of V_{j+1} for some (equivalently, for every) $j\in\mathbb{Z}$. By Lemma 10, it is sufficient to show that $\{(\psi_{\ell\delta})_{(-1)k}\}_{\ell\in J_n,\delta\in D,k\in\mathbb{Z}^n}$ is a basis of V_0 if and only if (4.25) holds.

First assume that $\{(\psi_{\ell\delta})_{(-1)k}\}_{\ell\in J_n,\delta\in D,k\in\mathbb{Z}^n}$ is a basis of V_0. Then $\{(\psi_{\ell\delta})_{(-1)k}\}_{\ell\in J_n\setminus\{0\},\delta\in D,k\in\mathbb{Z}^n}$ is a basis of W_{-1}. Then, (4.25) holds by Lemma 10.

Conversely, if (4.21) holds, then, by Remark 12, $((M_{\ell'}(\xi+\pi\alpha_{n,\ell''}))_{(\ell',\ell'')\in(J_n\setminus\{0\})\times J_n}$ is full rank. Hence, by Lemma 9, all the rows of this matrix form an orthonarmal basis of $\widetilde{W}_{-1}$ for almost all $\xi\in\mathbb{R}^n$. Define

$$\widehat{\Psi}_\ell(2\xi) := M_\ell(\xi)\widehat{\Phi}(\xi), \qquad \ell\in J_n\setminus\{0\}.$$

Then, $\{(\psi_{\ell\delta})_{(-1)k}\}_{\ell\in J_n\setminus\{0\},\delta\in D,k\in\mathbb{Z}^n}$ is an orthonormal basis of W_{-1}. □

Hereafter in this section, we assume that the given multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ is r-regular.

Notation 10. Denote by $E_j : L^2(\mathbb{R}^n) \longrightarrow V_j$, the orthogonal projection operator.

Then, as Meyer stated in [15, Section 2.10], we have the following two lemmas:

Lemma 11. *The orthogonal projection E_j of $L^2(\mathbb{R}^n)$ onto V_j can be represented as a pseudo-differential operator with the symbol: $\sigma(2^j x, 2^{-j}\xi)$, where*

$$\sigma(x,\xi) := \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\widehat{\Phi}(\xi)^*\widehat{\Phi}(\xi+2\pi k). \tag{4.26}$$

That is, for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$E_j f(x) = \int e^{ix\cdot\xi}\sigma(2^j x, 2^{-j}\xi)\widehat{f}(\xi)\,d\mu(\xi). \tag{4.27}$$

Proof. Since $\{\varphi_\delta(x-k)\}_{\delta\in D,k\in\mathbb{Z}^n}$ is an orthonormal basis of V_0 and φ_δ satisfies (c1) and (c2), then we can write

$$\begin{aligned}
E_0 f(x) &= \sum_{\delta\in D,k\in\mathbb{Z}^n} (f(x),\varphi_\delta(x-k))_{L^2(\mathbb{R}^n;dx)}\varphi_\delta(x-k) \\
&= \sum_{\delta\in D,k\in\mathbb{Z}^n} (\widehat{f}(\xi),\widehat{\varphi}_\delta(\xi)e^{-i\xi\cdot k})_{L^2(\mathbb{R}^n;d\mu(\xi))}\varphi_\delta(x-k) \\
&= \sum_{\delta\in D}\int e^{ix\cdot\xi}\widehat{f}(\xi)\overline{\widehat{\varphi}}_\delta(\xi)\sum_{k\in\mathbb{Z}^n} e^{-i(x-k)\cdot\xi}\varphi_\delta(x-k)\,d\mu(\xi).
\end{aligned}$$

By Poisson's summation formula, we have

$$\sum_{k\in\mathbb{Z}^n} e^{-i(x-k)\cdot\xi}\varphi_\delta(x-k) = \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\widehat{\varphi}_\delta(\xi+2\pi k).$$

Hence, property (b) in Definition 5 completes the proof. □

Lemma 12. *With the above notation,*

$$|\widehat{\Phi}(\xi)|^2 = 1 + O(|\xi|^{2r+2}), \quad \textit{as} \quad |\xi|\downarrow 0. \tag{4.28}$$

Proof. By the localization condition (c2), $\widehat{\Phi}(\xi)^*\widehat{\Phi}(\xi)$ is a smooth function. Since the oscillation condition (c4) imples that

$$(\partial_\xi^\alpha\widehat{\Phi})(0) = 0 \quad \text{for} \quad 1 \le |\alpha| \le 2r+1,$$

we have

$$((\partial_\xi^\beta\widehat{\Phi})^*(\partial_\xi^\gamma\widehat{\Phi}))(0) = 0 \quad \text{for} \quad 1 \le |\beta+\gamma| \le 2r+1.$$

A Taylor expansion implies that

$$\widehat{\Phi}(\xi)^*\widehat{\Phi}(\xi) = |\widehat{\Phi}(0)|^2 + O(|\xi|^{2r+2}), \quad \text{as} \quad |\xi|\downarrow 0.$$

Hence we need only show that $|\widehat{\Phi}(0)|^2 = 1$. The symbol (4.26) is so good that the pseudo-differential operator (4.27) can be applied to the constant function 1. In fact, we consider, for each $\varepsilon > 0$, the function $f_\varepsilon(x) := e^{-\varepsilon|x|^2}$ whose Fourier transform is $g_\varepsilon(\xi) := (\pi/\varepsilon)^{n/2}e^{-|\xi|^2/4\varepsilon}$. Then

$$E_0 f_\varepsilon(x) = \int e^{ix\cdot\xi}\sigma(x,\xi)g_\varepsilon(\xi)\,d\mu(\xi). \tag{4.29}$$

Denote

$$E(x,y) := \sum_{k\in\mathbb{Z}^n}\Phi(y-k)^*\Phi(x-k).$$

Then, since this $E(x,y)$ has the same properties as Meyer's $E(x,y)$ in [15, Section 2.6], it follows that Theorem 4 in [15, Section 2.6] is still valid for this $E(x,y)$, that is, we have

$$\int E(x,y)dy = 1.$$

Passing to the limit in (4.29), we have

$$1 = E_0 1 = \{e^{ix\cdot\xi}\sigma(x,\xi)\}_{\xi=0}, \qquad \text{in } \mathcal{S}'(\mathbb{R}^n). \tag{4.30}$$

This equality holds in $L^\infty(\mathbb{R}^n)$. Now, (4.26) gives

$$\sigma(x,0) = \sum_{k\in\mathbb{Z}^n} e^{2\pi ik\cdot x}\widehat{\Phi}(0)^*\widehat{\Phi}(2\pi k) = 1.$$

This implies that $\widehat{\Phi}(0)^*\widehat{\Phi}(0) = 1$ and $\widehat{\Phi}(0)^*\widehat{\Phi}(2\pi k) = 0$ for $k \neq 0$. □

5. Proof of Theorems 2 and 3

We shall prove Theorems 2 and 3 only in the case where $M_0(\xi) \in \mathrm{Mat}(d\times d; L^2(\mathbb{T}^n;\mathbb{C}))$ because the proof of the case where $M_0(\xi) \in \mathrm{Mat}(d\times d; L^2(\mathbb{T}^n;\mathbb{R}))$ is similar and rather easy.

Let us start with some general notation. Let $m(\xi) \in L^2(\mathbb{T}^n)$. The Fourier transform of $m(\xi)$ is

$$m(\xi) := \sum_{k\in\mathbb{Z}^n} \widehat{m}(k)e^{ik\cdot\xi}, \qquad \text{where } \widehat{m}(k) := \int_{\mathbb{T}^n} e^{-ik\cdot\xi}m(\xi)\,d\mu(\xi).$$

Put $k = 2l+\eta$ with k, $l \in \mathbb{Z}^n$ and $\eta \in R$. Since $\mathbb{Z}^n = 2\mathbb{Z}^n + R$, then

$$m(\xi) := \sum_{l\in\mathbb{Z}^n,\eta\in R} \widehat{m}(2l+\eta)e^{i(2l+\eta)\cdot\xi} = \sum_{\eta\in R} e^{i\eta\cdot\xi} \sum_{l\in\mathbb{Z}^n} \widehat{m}(2l+\eta)e^{i2l\cdot\xi}.$$

Notation 11. For $(m_{\ell'})_{(d',d'')}(\xi) \in L^2(\mathbb{T}^n)$, $\ell' \in J_n$, $d', d'' \in D$, denote

$$(m_{\ell'\ell''})_{(d',d'')}(\xi) := \sum_{l\in\mathbb{Z}^n} \widehat{(m_{\ell'})}_{(d',d'')}(2l+\alpha_{n,\ell''})e^{il\cdot\xi}, \quad \ell'' \in J_n. \tag{5.1}$$

Then,

$$(m_{\ell'})_{(d',d'')}(\xi) = \sum_{\ell''\in J_n} e^{i\alpha_{n,\ell''}\cdot\xi}(m_{\ell'\ell''})_{(d',d'')}(2\xi). \tag{5.2}$$

Since $(m_{\ell'\ell''})_{(d',d'')}(\xi)$ are $2\pi\mathbb{Z}^n$-periodic, we have, for $\eta \in R$,

(5.3)
$$\begin{aligned}(m_{\ell'})_{(d',d'')}(\xi+\eta\pi) &= \sum_{\ell''\in J_n} e^{i\alpha_{n,\ell''}\cdot(\xi+\eta\pi)}(m_{\ell'\ell''})_{(d',d'')}(2\xi)\\ &= \Big(2^{n/2}(m_{\ell'\ell''})_{(d',d'')}(2\xi)\Big)_{\ell''\in J_n} {}^t\Big(2^{-n/2}e^{i\alpha_{n,\ell''}\cdot(\xi+\eta\pi)}\Big)_{\ell''\in J_n}.\end{aligned}$$

Notation 12.

- $\breve{M}_{(d',d'')} := \big((m_{\ell'})_{(d',d'')}(\xi+\pi\alpha_{n,\ell''})\big)_{(\ell',\ell'')\in J_n\times J_n} \in \mathrm{Mat}(2^n\times 2^n; L^2(\mathbb{T}^n))$.
- $\breve{M}(\xi) := \big(\breve{M}_{(d',d'')}\big)_{(d',d'')\in D\times D} \in \mathrm{Mat}\big(d\times d; \mathrm{Mat}(2^n\times 2^n; L^2(\mathbb{T}^n))\big)$.
- $N_{(d',d'')} := \big(2^{n/2}(m_{\ell'\ell''})_{(d',d'')}(\xi)\big)_{(\ell',\ell'')\in J_n\times J_n} \in \mathrm{Mat}(2^n\times 2^n; L^2(\mathbb{T}^n))$.
- $N(\xi) := \big(N_{(d',d'')}\big)_{(d',d'')\in D\times D} \in \mathrm{Mat}\big(d\times d; \mathrm{Mat}(2^n\times 2^n; L^2(\mathbb{T}^n))\big)$.
- $U_{2^n}(\xi) := \big(2^{-n/2}e^{i\eta\cdot(\xi+r\pi)}\big)_{(\eta,r)\in R\times R} \in U(2^n; C^\infty(\mathbb{T}^n))$.
- $U(\xi) := \big(U_{2^n}\delta_{d',d''}\big)_{(d',d'')\in D\times D} \in U(2^n d; C^\infty(\mathbb{T}^n))$.

Remark 13. The matrix $\breve{M}(\xi)$ is given by changing the order of the rows of the matrix $\widetilde{M}(\xi)$ from the ordered set $J_n\times D$ with lexicographic order to the ordered set $D\times J_n$ with lexicographic order.

Since (5.3) is represented as $\breve{M}(\xi) = N(2\xi)U(\xi)$, then $\widetilde{M}(\xi)$ is unitary when $N(2\xi)$ is unitary. Hence, we have the following corollary to Proposition 4.

Corollary 1. *If $N(\xi)\in U(2^n d; L^2(\mathbb{T}^n))$, then the family of functions $\{\Psi_\ell\}_{\ell\in J_n\setminus\{0\}}$, defined by the relations $\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi)\widehat{\Phi}(\xi)$, is a family of wavelet functions.*

Lemma 13. *If the scaling fuction $\Phi(x)$ has the regularity (c1) and the localization property (c2) and if $N(\xi)\in U(2^n d; C^\infty(\mathbb{T}^n))$, then the family of functions $\{\Psi_\ell\}_{\ell\in J_n\setminus\{0\}}$, defined by the relations $\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi)\widehat{\Phi}(\xi)$, is a family of wavelet functions having the regularity (c1) and the localization property (c2).*

Proof. Assume that $N(\xi)\in U(2^n d; C^\infty(\mathbb{T}^n))$. Then, $\breve{M}(\xi)\in U(2^n d; C^\infty(\mathbb{T}^n))$ and, therefore, $M_\ell(\xi)\in \mathrm{Mat}(d\times d; C^\infty(\mathbb{T}^n))$ for $\ell\in J_n$.

Represent the elements of the matrix $M_\ell(\xi)$ by Fourier series as:

$$M_\ell(\xi) = \Big(\sum_{k\in\mathbb{Z}^n} \alpha_{\ell d' d'' k} e^{-ik\cdot\xi}\Big)_{(d',d'')\in D\times D}, \tag{5.4}$$

whose coefficients $\alpha_{\ell d' d'' k}$ are rapidly decreasing as $k\to\infty$. Then the relation $\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi)\widehat{\Phi}(\xi)$ implies

$$\begin{aligned} 2^{-n}\Psi_\ell(x/2) &= \Big(\sum_{k\in\mathbb{Z}^n} \alpha_{\ell d' d'' k}\Big)_{(d',d'')\in D\times D}\Phi(x-k) \\ &= \Big(\sum_{k\in\mathbb{Z}^n, d''\in D} \alpha_{\ell d' d'' k}\varphi_{d''}(x-k)\Big)_{d'\downarrow 1,\dots,d}. \end{aligned} \tag{5.5}$$

Diffentiating (5.5) under the summation, we can show that every Ψ_ℓ, $\ell\in J_n\setminus\{0\}$, has the same regularity and localization property as Φ. □

Proof of Theorem 2. By Corollary 1, it suffices to construct $N(\xi)\in U(2^n d; L^2(\mathbb{T}^n))$. Since a multiresolution analysis is given, that is, $M_0(\xi)$ is given, then d rows

$$\{2^{n/2}(m_{0\ell''})_{(d',d'')}(\xi)\}_{(d'',\ell'')\in D\times J_n}, \quad d'\in D \tag{5.6}$$

of $N(\xi)$ are given, which is an orthonormal system in $\mathbb{C}^{2^n d}$ for almost all $\xi\in\mathbb{R}^n$. We must construct the remaining $(2^n-1)d$ rows

$$\{2^{n/2}(m_{\ell'\ell''})_{(d',d'')}(\xi)\}_{(d'',\ell'')\in D\times J_n}, \qquad (d',\ell')\in D\times J_n, \tag{5.7}$$

of $N(\xi)$ so that $N(\xi) \in U(2^n d; L^2(\mathbb{T}^n))$. Using the Gram-Schmidt orthonormalization for every $\xi \in \mathbb{T}^n$, we can construct $N(\xi) \in U(2^n d; L^2(\mathbb{T}^n))$. This completes the proof. □

Remark 14. If the scaling fuction has the localiation property (c2), then we can apply Theorem 1 to the construction of $N(\xi) \in U(2^n d; C^\infty(\mathbb{T}^n))$ as we stated in Remark 2. Then we have wavelet functions which have the localization property (c2) by Lemma 13.

Lemma 14. *Let $N(\xi) \in U(2^n d; L^2(\mathbb{T}^n))$ be given. Then*

$$\sum_{\delta\in D} |\widehat{\varphi}_\delta(\xi)|^2 + \sum_{\delta\in D,\ell\in J_n\backslash\{0\}} |\widehat{\psi}_{\ell\delta}(\xi)|^2 = \sum_{\delta\in D} |\widehat{\varphi}_\delta(\xi/2)|^2, \qquad \text{a.a. } \xi. \tag{5.8}$$

Proof. Denote

$$M^\circ(\xi) := \big(M_\ell(\xi); \ell \downarrow 0, \dots, 2^n - 1\big) \in \mathrm{Mat}(2^n d \times d; L^2(\mathbb{T}^n)). \tag{5.9}$$

Since $N(\xi) \in U(2^n d; L^2(\mathbb{T}^n))$ is given, then we have $M^\circ(\xi)$ whose columns are orthonormal, that is,

$$M^\circ(\xi)^* M^\circ(\xi) = I_d. \tag{5.10}$$

Multiply both sides of (5.10) by $\widehat{\Phi}(\xi)^*$ from the left and by $\widehat{\Phi}(\xi)$ from the right. Then

$$(M_\ell(\xi)\widehat{\Phi}(\xi); \ell \downarrow 0, \dots, 2^n - 1)^* (M_\ell(\xi)\widehat{\Phi}(\xi); \ell \downarrow 0, \dots, 2^n - 1) = \widehat{\Phi}(\xi)^* \widehat{\Phi}(\xi).$$

Since $M_\ell(\xi)\widehat{\Phi}(\xi) = \widehat{\Psi}_\ell(2\xi)$, then we have

$$(\widehat{\Psi}_\ell(2\xi); \ell \downarrow 0, \dots, 2^n - 1)^* (\widehat{\Psi}_\ell(2\xi); \ell \downarrow 0, \dots, 2^n - 1) = \widehat{\Phi}(\xi)^* \widehat{\Phi}(\xi),$$

which is the conclusion sought. □

Now we can prove Theorem 3.

Proof of Theorem 3. We use the same construction as Remark 14. Lemma 13 ensures that the family of wavelet functions constructed as above has the regularity (c1) and the localization property (c2). To establish the oscillation property (c3), we substitute (4.28) in (5.8); thus we have

$$\sum_{\delta\in D,\ell\in J_n\backslash\{0\}} |\widehat{\psi}_{\ell\delta}(\xi)|^2 = O(|\xi|^{2r+2}), \quad \text{as} \quad |\xi| \downarrow 0,$$

that is,

$$|\widehat{\psi}_{\ell\delta}(\xi)| = O(|\xi|^{r+1}), \qquad \text{as } |\xi| \downarrow 0, \qquad \text{for } \delta \in D,\ \ell \in J_n\backslash\{0\}.$$

Hence $(\partial^\alpha \widehat{\psi}_{\ell\delta})(0) = 0$, for $\delta \in D$ and $\ell \in J_n\backslash\{0\}$. Thus (c3) holds. The proof is complete. □

References

1. B. Alpert, *A class of bases in L^2 for the sparse representation of integral operators*, SIAM J. Math. Anal. **24** (1993), 246–262.
2. C. de Boor, R. DeVore and A. Ron, *On the construction of multivariate (pre) wavelets*, Constr. Approx. **9** (1993), 123–166.
3. C. K. Chui, J. Stöckler and J. D. Ward, *Compactly supported box spline wavelets*, Approx. Theory Appl. **8** (1992), 77–100.
4. I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF regional conference series in applied mathematics, vol. 61, SIAM, Philadelphia, 1992.
5. R. DeVore and and B Lucier, *Wavelets*, Acta Numerica **1** (1991), 1–56.
6. T.N.T. Goodman, S.L. Lee and Z. Shen, *Wavelets of multiplicity r*, Trans. Amer. Math. Soc. **342** (1994), 307–324.
7. T.N.T. Goodman, S.L. Lee and W.S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc. **338** (1993), 639–654.
8. G. K. Gröchenig, *Analyse multi-échelle et bases d'ondelettes*, C. R. Acad. Sci. Paris, Série I **305** (1987), 13–15.
9. D. Hardin, D. Kessler and P.R. Massopust, *Multiresolution analysis based on fractal functions*, J. Approx. Theory **71** (1992), 104–120.
10. L. Herve, *Méthodes d'opérateurs d'échelle généralisées et ondelettes à support compact*, Revista Mat. Iberoamer **9** (1993), 333–31.
11. R. Q. Jia and C.A. Micchelli, *Using the refinement equation for the construction of pre-wavelets II: Powers of two*, Curves and Surfaces, (P. J. Laurent, A. Le Méhauté and L. L. Schumaker, eds) (1991), Academic Press, New York, 1–56.
12. R. Q. Jia and C.A. Micchelli, *Using the refinement equation for the construction of pre-wavelets V: extensibility of trigonometric polynomials*, Computing **48** (1992), 61–72.
13. R.-Q. Jia and Z. Shen, *Multiresolution and wavelets*, Proc. Edinburgh Math. Soc. **37** (1994), 271–300.
14. W. Lawton, S.L. Lee and Z. Shen, *An algorithum for matrix extension and wavelet construction*, Math. Comp. (to appear).
15. Y. Meyer, *Wavelets and Operators*, Cambridge Studies in Advanced Mathematics, vol. 37, Cambridge University Press, Cambridge, 1992.
16. S. Sternberg, *Lecture on Differential Geometry*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
17. G. Strang and V. Strela, *Short wavelets and matrix dilation equations*, submitted to IEEE Transactions on Signal Processing (1993).
18. G. Strang and V. Strela, *Orthogonal multiwavelets with vanishing moments*, Optical Engineering (to appear).