Image restoration through microlocal analysis with smooth tight wavelet frames*

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Abstract

General results on frame, microlocal analysis, and tight frame wavelets in $\mathbb{R}^n$ are summarized. To perform microlocal analysis of tempered distributions in $\mathbb{R}^n$, tight frame wavelets, whose Fourier transforms consist of smoothed characteristic functions of squares or sectors of annuli in $\mathbb{R}^2$ and cubes in $\mathbb{R}^3$, are constructed. Singularities in smooth images are localized in position and direction by means of the frame coefficients computed in the Fourier domain by using Plancherel's theorem.

1 Introduction

In previous work [2], [3], [4], the concept of microlocal analysis was described in view of studying the singularities of distributions in $\mathbb{R}^n$. Initially, the

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microlocal analytic content of a distribution was localized by means of the coefficients of a multiwavelet expansion. The Fourier transform of the multiwalets were characteristic functions of boxes or squares that completely covered the Fourier domain. Next, to have better resolution in the x domain, smooth tight wavelet frames were constructed by different smoothings the box functions [5]. A numerically convenient smoothing was done by tapering the characteristic functions in $\mathbb{R}^2$ [6]. Image analysis in the Fourier domain allows the localization of the microanalytic content and singularities in position and orientation. It also allows some denoising and compression of natural and geometric images.

In this paper, a further step is taken in restoring images by removing singularities that have been localized by the above methods. The theoretical results have to be adapted to finite images and direct and inverse discrete Fourier transforms. A simple example is treated where a singularity in the form of a straight segment is added to a natural image, called the original image, A, thus producing a scarred image, S. The discrete Fourier transform of the scarred image, FS, is filtered by means of a wavelet frame with support in the high frequency part of the transformed image in order to pick up the singularity. This high pass filter, H removes the low frequencies which contain the Fourier transform of the original image A. This amounts to compute the frame coefficients of the transformed image FS. By means of the Plancherel theorem, the coefficients with larger absolute values localize the scar in the x domain. The scar is reconstruted by means of its wavelet frame expansion in the x domain. Because of its finite size, the one-pixel thick scar is returned to the x domain as a few pixel thick line or segment after a direct and an inverse discrete Fourier transform. To remove small perturbations in the returned image, the values at each pixel is rounded to set the small values to zero. Then the thickness of the line or segment is found and reduced by setting to zero the pixels off the center line. Then subtracting this image from the initial scarred image produces the original image.

2 Frames

We briefly review frame theory based on Mallat [14]. Frame theory was originally developed by Duffin and Schaeffer [7] to reconstruct band-limited signals f from irregularly spaced samples $\{f(t_n)\}_{n\in\mathbb{Z}}$. A function f is said to be band limited if its Fourier transform is supported in a finite interval $[-\pi/T, \pi/T]$, Duffin and Schaeffer were motivated by the classical Shanonn sampling theorem, which asserts that

$$f(t_n) = \frac{1}{T} \langle f(t), h_T(t - t_n) \rangle, \qquad h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

They discussed general conditions under which one can recover a vector f in a separable Hilbert space $\mathcal{H}$ from its inner products $\langle f, \phi_n \rangle$ with a family of vectors $\{\phi_n\}_{n\in\mathbb{J}}$, where the index set $\mathbb{J}$ might be finite or infinite.

A sequence $\{\phi_n\}_{n\in\mathbb{J}}$ is called a *frame* of $\mathcal{H}$ if there exist two constants $A > 0$ and $B > 0$ such that for any $f \in \mathcal{H}$

$$A \|f\|^2 \le \sum_{n\in\mathbb{J}} |\langle f, \phi_n \rangle|^2 \le B \|f\|^2.$$

The constants A and B are called *frame bounds*. A frame is said to be *tight* if $A = B$. The operator $L : \mathcal{H} \mapsto \mathcal{H}$ defined by

$$Lf = \sum_{n\in\mathbb{J}} \langle f, \phi_n \rangle \phi_n, \qquad \forall f \in \mathcal{H}$$

is called the *frame operator*. Denote

$$\ell^2(\mathbb{J}) := \{x \ : \ \|x\|^2_{\ell^2(\mathbb{J})} := \sum_{n\in\mathbb{J}} |x[n]|^2 < +\infty\}.$$

The definition of frame gives an energy equivalence to invert the operator $U : \mathcal{H} \mapsto \ell^2(\mathbb{J})$ defined by

$$Uf[n] = \langle f, \phi_n \rangle, \qquad \forall n \in \mathbb{J}.$$

Denote by U^* the *adjoint* of U: $\langle Uf, x \rangle = \langle f, U^* x \rangle$. Then the frame operator L can be represented as

$$Lf = U^*Uf = \sum_{n\in\mathbb{J}} \langle f, \phi_n \rangle \phi_n.$$

The system $\{\tilde{\phi}_n\}_{n\in\mathbb{J}}$ defined by

$$\tilde{\phi}_n = L^{-1}\phi_n \ = (U^*U)^{-1}\phi_n$$

is called the *dual frame* of $\{\phi_n\}_{n\in\mathbb{J}}$. If the frame is tight (i.e., $A = B$), then $\tilde{\phi}_n = A^{-1} \phi_n$. The dual frame satisfies

$$\frac{1}{B} \|f\|^2 \le \sum_{n\in\mathbb{J}} |\langle f, \tilde{\phi}_n \rangle|^2 \le \frac{1}{A} \|f\|^2, \qquad \forall f \in \mathcal{H}.$$

Denote by $\operatorname{ran} U$ the range of U, that is, the space of all Uf with $f \in \mathcal{H}$. If $\{\phi_n\}_{n\in\mathbb{J}}$ is a frame whose vectors are linearly dependent, then $\operatorname{ran} U$ is strictly included in $\ell^2(\mathbb{J})$, and U admits an infinite number of left inverses $\bar{U}^{-1}$:

$$\bar{U}^{-1}Uf = f, \qquad \forall f \in \mathcal{H}.$$

The left inverse that is zero on $\operatorname{ran} U^{\perp}$ is called the *pseudo inverse* and denoted by $\tilde{U}^{-1}$:

$$\tilde{U}^{-1}x = 0, \qquad \forall x \in \operatorname{ran} U^{\perp}.$$

In infinite dimensional spaces, the pseudo inverse $\tilde{U}^{-1}$ of an injective operator is not necessarily bounded. This induces numerical instabilities when trying to reconstruct f from Uf. The pseudo inverse satisfies

$$\tilde{U}^{-1} = (U^*U)^{-1}U^*$$

and

$$f = \tilde{U}^{-1}Uf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle \, \tilde{\phi}_n = \sum_{n\in\mathbb{J}} \langle f, \tilde{\phi}_n\rangle \, \phi_n.$$

When the frame is tight (i.e., $A = B$), as $\tilde{\phi}_n = A^{-1}\,\phi_n$,

$$f = \tilde{U}^{-1}Uf = \frac{1}{A}\sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle \, \phi_n.$$

In this case, we may assume the frame bound $A = 1$ without loss of generality by replacing ϕ_n by $\phi_n/\sqrt{A}$.

Let us describe efficient numerical algorithms to recover a signal f from its frame coefficients $Uf[n] = \langle f, \phi_n\rangle$. When the dual frame vectors $\tilde{\phi}_n = (U^*U)^{-1}\phi_n$ are precomputable, we can recover each f with the sum

$$f = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle \, \tilde{\phi}_n.$$

But in some applications, the frame vectors $\{\phi_n\}_{n\in\mathbb{J}}$ may depend on the signal f, in which case the dual frame vectors $\tilde{\phi}_n$ cannot be computed in advance. It is not efficient to compute the dual frame for each new signal. A more direct approach applies the pseudo inverse to Uf:

$$f = \tilde{U}^{-1}Uf = (U^*U)^{-1}(U^*U)f = L^{-1}Lf.$$

Whether we precompute the dual frame vectors or apply the pseudo inverse on the frame data, both approaches require an efficient way to compute $f = L^{-1}g$ for some $g \in \mathcal{H}$. One way is to use the following *Richardson's extrapolation* when the frame bounds A and B are known.

Lemma 1 (Richardson's Extrapolation) *Let $g \in \mathcal{H}$. To compute $f = L^{-1}g$ we initialize $f_0 = 0$. Let $\gamma > 0$ be a relaxation parameter. For any $n > 0$, define*

$$f_n = f_{n-1} + \gamma\,(g - Lf_{n-1}).$$

If

$$\delta = \max\{|1-\gamma A|, |1-\gamma B|\} < 1,$$

then

$$\|f - f_n\| \leq \delta^n\,\|f\|, \tag{1}$$

and hence $\lim_{n\to+\infty} f_n = f$.

For frame inversion, Daubechies [8] called this Richardson's extrapolation algorithm as the *frame algorithm*. The convergence rate is maximized when δ is minimum:

$$\delta = \frac{B-A}{B+A} = \frac{1-A/B}{1+A/B},$$

which corresponds to the relaxation parameter

$$\gamma = \frac{2}{A+B}.$$

The algorithm converges quickly if A/B is close to 1. If A/B is small then

$$\delta \approx 1 - 2\,\frac{A}{B}. \tag{2}$$

Inequality (1) proves that we obtain an error smaller than ϵ for a number n of iterations, which satisfies:

$$\frac{\|f - f_n\|}{\|f\|} \leq \delta^n = \epsilon.$$

Inserting (2) gives

$$n \approx \frac{\log\epsilon}{\log(1-2A/B)} \approx \frac{-B}{2A}\log\epsilon.$$

Thus, the number of iterations is directly proportional to the frame bound ratio B/A.

3 Tight Wavelet Frames

Since the dual of a tight frame is a constant multiple of the frame itself, recovering functions from their frame coefficients does not require the calculation of the dual frame. Hereafter, we shall focus on tight wavelet frames.

Given $f \in L^2(\mathbb{R}^n)$, let $f_{jk}(x)$ denote the scaled and shifted function

$$f_{jk}(x) = 2^{nj/2} f(2^j x - k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n. \tag{3}$$

Let $\mathbb{L}$ be a finite index set. A system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called a *tight wavelet frame* with frame bound A if

$$f(x) = \frac{1}{A} \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \langle f, \psi^\ell_{jk}\rangle\, \psi^\ell_{jk}(x), \qquad \forall f \in L^2(\mathbb{R}^n). \tag{4}$$

We recall that a system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called an *orthonormal wavelet basis* if it is an orthonormal basis for $L^2(\mathbb{R}^n)$. This is equivalent to saying that the system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n}$ is a tight wavelet frame with frame bound 1 and $\|\psi^\ell\|_{L^2(\mathbb{R}^n)} = 1$ for $\ell \in \mathbb{L}$.

The following general theorem which is essentially Theorem 1 stated and proved in [13] for $\mathbb{R}^n$, gives necessary and sufficient conditions to have a tight wavelet frame in $\mathbb{R}^n$ with frame bound 1.

Theorem 1 *Suppose $\psi^\ell \subset L^2(\mathbb{R}^n)$ for $\ell \in \mathbb{L}$, then*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \left|\langle f, \psi^\ell_{j,k}\rangle\right|^2 \tag{5}$$

for all $f \in L^2(\mathbb{R}^n)$ if and only if the set of functions $\{\psi^\ell\}_{\ell\in\mathbb{L}}$ satisfies the following two equalities:

$$\sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}}} \left|\widehat{\psi}^\ell(2^j\xi)\right|^2 = 1, \qquad a.e.\, \xi \in \mathbb{R}^n, \tag{6}$$

$$\sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}_+}} \widehat{\psi}^\ell(2^j\xi)\overline{\widehat{\psi}^\ell(2^j(\xi+q))} = 0, \qquad a.e.\, \xi \in \mathbb{R}^n,\ \forall q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n, \tag{7}$$

where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n$ means that at least one component q_j is odd.

Corollary 1 *Under the hypotheses of Theorem 1, any function $f \in L^2(\mathbb{R}^n)$ admits the tight wavelet frame expansion*

$$f(x) = \sum_{\substack{\ell \in \mathbb{L} \\ j \in \mathbb{Z} \\ k \in \mathbb{Z}^n}} \langle f, \psi^{\ell}_{jk} \rangle \, \psi^{\ell}_{jk}(x). \tag{8}$$

By using the localization property of the frame wavelet in the Fourier domain, one can study the directions of growth of $\widehat{f}(\xi)$ by looking at the size of the frame coefficients

$$\langle f, \psi^{\ell}_{jk} \rangle = (2\pi)^{-n} \langle \widehat{f}, \widehat{\psi}^{\ell}_{jk} \rangle, \tag{9}$$

where the Fourier transform of f is defined by

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} \, f(x) \, dx$$

and the inverse Fourier transform of g is defined by

$$\mathcal{F}^{-1}[g](x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} \, g(\xi) \, d\xi.$$

Moreover, by using the localization property of the frame wavelets in x-space, one can localize the singular support of $f(x)$ by varying ℓ, j and k in (9).

4 Frame Multiresolution Analysis

The notion of frame multiresolution analysis was introduced by Benedetto and Li [1]. Let us recall that an *(orthonormal) multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying

(i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;

(ii) $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$;

(iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

(iv) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(v) There exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}^n}$ is an orthonormal basis for V_0.

The function $\varphi \in L^2(\mathbb{R}^n)$ whose existence is asserted in the condition (v) is called an *(orthonormal) scaling function* of the given orthonormal multiresolution analysis.

A *frame multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and (v-1) There exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^n}$ is a frame for V_0.

The function $\varphi \in L^2(\mathbb{R}^n)$ whose existence is asserted in the condition (v-1) is called a *frame scaling function* of the given frame multiresolution analysis.

Let D be a finite index set. An *(orthonormal) multiresolution analysis for multiwavelets* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-2) There exists a system of functions $\{\varphi_\delta\}_{\delta\in D} \subset V_0$ such that $\{\varphi_\delta(\cdot - k)\}_{\delta\in D,\ k\in\mathbb{Z}^n}$ is an orthonormal basis for V_0.

The functions $\{\varphi_\delta\}_{\delta\in D}$ whose existence is asserted in the condition (v-2) are called *(orthonormal) multiscaling functions*.

A *frame multiresolution analysis for multiwavelet* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-3) There exists a system of functions $\{\varphi_\delta\}_{\delta\in D} \subset V_0$ such that $\{\varphi_\delta(\cdot - k)\}_{\delta\in D,\ k\in\mathbb{Z}^n}$ is a frame for V_0.

The functions $\{\varphi_\delta\}_{\delta\in D}$ whose existence is asserted in the condition (v-3) are called *frame multiscaling functions*.

5 Microlocal Analysis

Our approach to microlocal analysis is based on the theory of hyperfunctions ([10], [11], [12]). They are powerful tools in several applications; for example, vortex sheets in two-dimensional fluid dynamics are a realization of hyperfunctions of one variable. Microlocal analysis deals with the direction along which a hyperfunction can be extended analytically. In other words, it decomposes the "singularity" into microlocal directions. Microlocal analysis plays an important role in the theory of hyperfunctions, partial differential operators, and many other areas. In this theory, for example, one can consider the product of hyperfunctions and discuss the partial regularity of hyperfunctions with respect to any independent variable.

Here, we give only a rough sketch. A more complete treatment of microlocal filtering can be found in R. Ashino, C. Heil, M. Nagase, and R. Vaillancourt [2] (See also [5]). The important point is to find directions in which a hyperfunction can be continued analytically. Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\Gamma \subset \mathbb{R}^n$ be a convex open cone with vertex at 0. From now on, every cone is assumed to have vertex at 0. The set $\Omega + i\Gamma \subset \mathbb{C}^n$ is called a *wedge*.

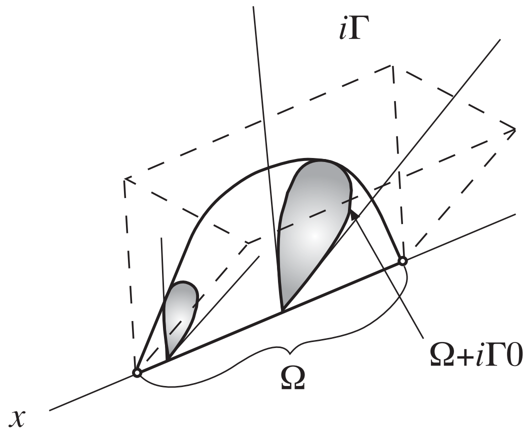

Figure 1: An infinitesimal wedge $\Omega + i\Gamma 0$.

An *infinitesimal wedge* $\Omega + i\Gamma 0$ is an open set $U \subset \Omega + i\Gamma$ which approaches asymptotically to Γ as the imaginary part tends to 0. (Figure 1.)

A *hyperfunction* $f(x)$ can be defined as a sum

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \qquad x \in \Omega,$$

of formal boundary values

$$F_j(x + i\Gamma_j 0) = \lim_{\substack{y \to 0 \\ x+iy \in \Omega + i\Gamma_j 0}} F_j(x + iy)$$

of holomorphic functions $F_j(z)$ in the infinitesimal wedges $\Omega + i\Gamma_j 0$.

A hyperfunction is said to be *micro-analytic* in the direction $\xi_0 \in \mathbb{S}^{n-1}$ at $x_0 \in \mathbb{R}^n$ or in short, at (x_0, ξ_0), if there exists a neighborhood Ω of x_0 and holomorphic functions F_j on infinitesimal wedges $\Omega + i\Gamma_j 0$ such that $f = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0)$ and

$$\Gamma_j \cap \{y \in \mathbb{R}^n \ : \ y \cdot \xi_0 < 0\} \neq \emptyset$$

for all j.

A simple aspect of the relation between micro-analyticity and Fourier transform is given as follows.

Lemma 2 *Let $\Gamma \subset \mathbb{R}^n$ be a closed cone and $x_0 \in \mathbb{R}^n$. For a tempered distribution f, if there exists a tempered distribution g such that* $\operatorname{supp} \hat{g} \subset \Gamma$ *and $f - g$ is analytic in a neighborhood of x_0, then f is micro-analytic at (x_0, ξ) for every $\xi \in \Gamma^c \cap \mathbb{S}^{n-1}$, where Γ^c denotes the complement of Γ.*

Our aim of this paper is to answer to the following questions:

- Is it possible to construct orthonormal or tight frame multiwavelets $\Psi = \{\psi^\delta\}_{\delta\in D}$ corresponding to each microanalytic direction $\xi \in \mathbb{S}^{n-1}$?
- Is it possible to obtain information on the microlocal content of $f \in L^2(\mathbb{R}^n)$ from the wavelet coefficients $\langle f, \psi^\delta_{jk}\rangle$?
- Can orthonormal or tight frame multiwavelet filtering separate microlocal contents?

We shall construct orthonormal multiwavelet bases or tight frames which enable us to obtain information on the microlocal content of signals or functions. Since this separation of microlocal contents can be considered as a filtering operation, we call it *microlocal filtering*.

6 One-dimensional Orthonormal Microlocal Filtering

Our aim is to construct wavelets $\{\phi_\delta\}_{\delta\in D}$ having good localization both in the base space $\mathbb{R}$ and in the direction space $\mathbb{S}^0 = \{\pm 1\}$ within the limits of the uncertainty principle. Here good localization at a point $(x_0, \xi_0) \in \mathbb{R}\times\mathbb{S}^0$, which is called *good microlocalization*, means that ϕ_δ is essentially concentrated in a neighborhood of a point $x_0 \in \mathbb{R}$ and $\hat{\phi}_\delta$ is essentially concentrated in a conic neighborhood of a point $\xi_0 \in \mathbb{S}^0$.

Define the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ by

$$H^2(\mathbb{R}_\pm) = \big\{ f \in L^2(\mathbb{R}) : \ \widehat{f}(\xi) = 0 \quad \text{a.e. } \xi \le (\ge)\, 0 \big\}.$$

Each function of the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ has good localization in the direction space $\mathbb{S}^0 = \{\pm 1\}$. Hence if we construct wavelets the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ having good localization in the base space, those wavelets have good microlocalization.

In these examples, an orthonormal wavelet function ψ_+ and an orthonormal scaling function ϕ_+ for orthonormal wavelets of $H^2(\mathbb{R}_+)$ are defined by

$$\widehat{\psi}_+ = \chi_{[2\pi, 4\pi]}, \qquad \widehat{\phi}_+ = \chi_{[0,2\pi]}.$$

From the two-scale relation

$$2\widehat{\phi}_+(2\xi) = m_0(\xi)\widehat{\phi}_+(\xi)$$

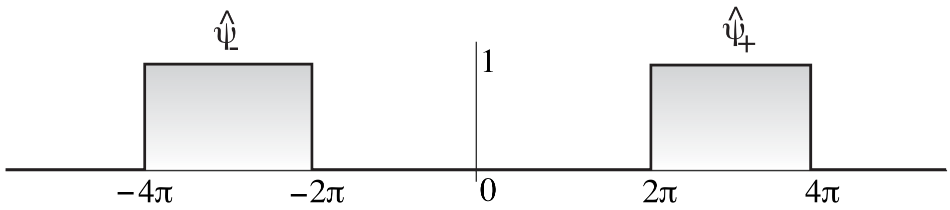

Figure 2: The Fourier transform of the orthonormal wavelet functions ψ_+ and ψ_-.

it is found that the corresponding lowpass filter is

$$m_0(\xi) = 2\chi_{[0,\pi]}(\xi) = 2\widehat{\phi}_+(2\xi)$$

on $[0, 2\pi)$, and extended 2π-periodically. From the two-scale relation

$$2\widehat{\psi}_+(2\xi) = e^{i\xi}\,\overline{m_0(\xi+\pi)}\widehat{\phi}_+(\xi) = m_1(\xi)\widehat{\phi}_+(\xi)$$

it is found that the corresponding highpass filter is

$$m_1(\xi) = e^{i\xi}\,\overline{m_0(\xi+\pi)} = 2\widehat{\psi}_+(2\xi)$$

on $[0, 2\pi)$, and extended 2π-periodically.

By the same argument, we have an orthonormal wavelet function ψ_- and an orthonormal scaling function ϕ_- for orthonormal wavelets of $H^2(\mathbb{R}_-)$. Since

$$L^2(\mathbb{R}) = H^2(\mathbb{R}_+) \oplus H^2(\mathbb{R}_-),$$

$\{\psi_+, \psi_-\}$ and $\{\phi_+, \phi_-\}$ can be regarded as orthonormal multiwavelet functions and orthonormal multiscaling functions, respectively, of $L^2(\mathbb{R})$. This decomposition of $L^2(\mathbb{R})$ into the orthogonal sum of the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ corresponds to the intuitive definition of hyperfunction:

$$f(x) = F_+(x+i0) - F_-(x-i0),$$

where $F_+(z)$ and $F_-(z)$ are holomorphic in the upper half plane and in the lower half plane, respectively.

It must be remarked that Auscher [9] essentially proved that there is no smooth orthonormal wavelet ψ in the classical Hardy space $H^2(\mathbb{R}_+)$, that is, there is no orthonormal wavelet ψ whose Fourier transform $\widehat{\psi}$ is continuous on $\mathbb{R}$ and satisfies the regularity condition:

$$\exists\alpha > 0; \qquad |\widehat{\psi}(\xi)| = O\big((1+|\xi|)^{-\alpha-1/2}\big) \quad \text{at } \infty.$$

The decay of a function at infinity in x space corresponds to the smoothness of its Fourier transform in ξ space. Hence the non-existence of smooth wavelets implies that it is impossible to have any smooth orthonormal wavelet having good microlocalization. Thus our aim comes to the construction of smooth tight frame wavelets having good microlocalization.

7 Multi-dimensional Orthonormal Microlocal Filtering

The following notation will be used.

- $\eta = (\eta_1, \ldots, \eta_n) \in H := \{\pm 1\}^n$.
- $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E := \{0, 1\}^n \setminus \{0\}, \quad j \in \mathbb{Z}_+$.
- $Q_\eta := \prod_{k=1}^n [0, \eta_k], \quad \varepsilon . * \eta := (\varepsilon_1\eta_1, \ldots, \varepsilon_n\eta_n)$.
- $\mathcal{Q}_{j,\varepsilon,\eta} := \big\{\prod_{k=1}^n [\eta_k(\ell_k - 1), \eta_k\ell_k] + 2^j(\varepsilon . * \eta) : 1 \leq \ell_1, \ldots, \ell_n \leq 2^j, \ \ell_1, \ldots, \ell_n \in \mathbb{N}\big\}$.
- $\mathbb{Z}_+^{E\times H}$ is the set of all functions from $E \times H$ to $\mathbb{Z}_+$.

Theorem 2 *Fix $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$. For a cube $Q \in \mathcal{Q}_{j,\varepsilon,\eta}$, define ψ_Q by*

$$\widehat{\psi}_Q = \chi_{2\pi Q},$$

where $\chi_{2\pi Q}$ is the characteristic function of the cube $2\pi Q$. For $\rho \in \mathbb{Z}_+^{E\times H}$, let

$$\mathcal{Q}_\rho := \bigcup_{(\varepsilon,\eta)\in E\times H} \mathcal{Q}_{\rho(\varepsilon,\eta),\varepsilon,\eta}.$$

Then $\Psi := \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ is a set of orthonormal wavelets. Define φ_η by

$$\widehat{\varphi}_\eta := \chi_{2\pi Q_\eta}.$$

Then $\{\varphi_\eta\}_{\eta\in H}$ are frame scaling functions for these wavelets.

In particular, when $\rho(\varepsilon, \eta)$ is constant, Ψ are multiwavelets.

Figure 3 illustrates the 2-D multiwavelets constructed in Theorem 2. Multiwavelets are masks in Fourier space — they are characteristic functions of cubes $2\pi Q$. The left part of Fig. 3 shows 12 multiwavelet functions. For finer resolution in Fourier space, we need a greater number of multiwavelets. The right part of Fig. 3 shows 27 multiwavelet functions.

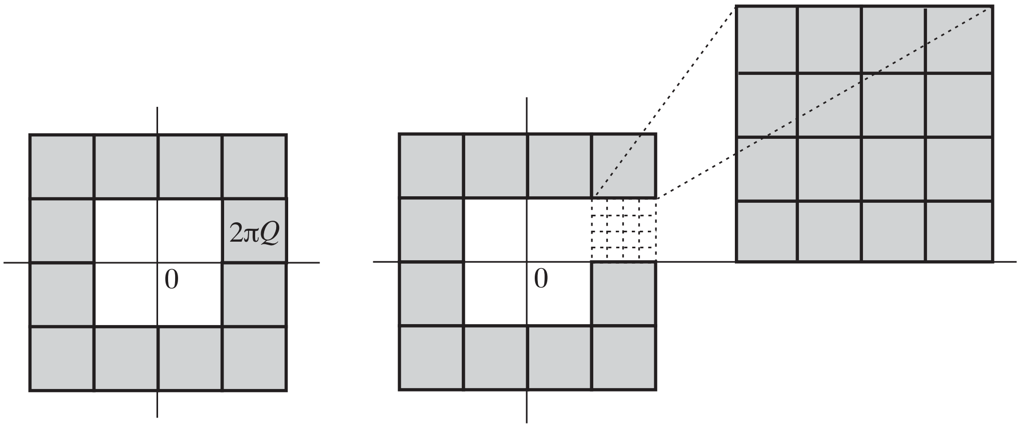

Figure 3: 2-D orthonormal multiwavelet functions in Fourier space.

8 Multi-dimensional Frame Microlocal Filtering

Smooth tight multiwavelet frames are obtained by convolving characteristic functions of cubes πQ so that the support of the smoothed functions have support inside cubes $2\pi Q$. This is achieved by considering the next inside annulus of cubes πQ in the left part of Fig. 3.

Let $\vartheta(t)$ be a $C_0^\infty(\mathbb{R})$-function of one variable satisfying

$$\vartheta(t) \geq 0, \qquad \vartheta(t) = \vartheta(-t), \qquad \int_{\mathbb{R}} \vartheta(t)\,dt = 1, \qquad \vartheta(t) = \begin{cases} 1, & |t| \leq \frac{1}{3}; \\ 0, & |t| \geq \frac{2}{3}. \end{cases}$$

For $\alpha > 0$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, let

$$\vartheta_\alpha(\xi) = \frac{1}{\alpha^n} \prod_{j=1}^{n} \vartheta\Big(\frac{\xi_j}{\alpha}\Big).$$

Theorem 3 *Fix $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$, and $\alpha \in (0, 1/2)$. Define*

$$\lambda_Q(\xi) := (\vartheta_\alpha * \chi_{\pi Q})(\xi) = \int_{\mathbb{R}^n} \vartheta_\alpha(\xi - \zeta)\,\chi_{\pi Q}(\zeta)\,d\zeta, \qquad Q \in \mathcal{Q}_{j,\varepsilon,\eta},$$

where $\chi_{\pi Q}$ is the characteristic function of the cube πQ. For $\rho \in \mathbb{Z}_+^{E\times H}$, let

$$\tau_\rho(\xi) := \sum_{j\in\mathbb{Z}, Q\in\mathcal{Q}_\rho} |\lambda_Q(2^j\xi)|^2,$$

and, for $Q \in \mathcal{Q}_\rho$*, define* $\psi_Q(x)$ *by*

$$\widehat{\psi}_Q(\xi) := \tau_\rho(\xi)^{-1/2}\,\lambda_Q(\xi).$$

Then $\Psi := \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ *is a set of tight frame wavelets.*

Theorem 3 follows from Theorem 1.

9 Numerical Restoration of Images

In this section, we apply the above theory to the restoration of finite images represented by matrices. Since the Fourier transform of a finite region gives rise to oscillations of the type of cardinal sine, care must be taken in the restoration process.

The restoration process involves the following steps.

- The figure A to be restored is Fourier transformed into B.
- B is filtered by multiplication with a tapered characteristic function with support far from the origin and at right angle with the singularity to be localized. This produces C.
- In view of the Plancherel theorem, the wavelet coefficients of C,

  $$\langle \widehat{f}, \widehat{\psi}^{\ell}_{jk}\rangle = \text{const.}\langle f, \psi^{\ell}_{jk}\rangle,$$

  are constructed in the Fourier domain and used in the x domain, to produce D which is the wavelet frame expansion (8) of Corollary 1.
- The extra width of D, caused by the side lobes in the support of ψ^{ℓ}_{jk}, is narrowed to eliminate oscillations due the cardinal sine effect when transforming functions with finite support.
- A tuned multiple of D is subtracted from A to restore the original image E.

In Fig. 4, the scarred woman image is restored. One notices in the top right part of the figure the wide width of the negative of the Fourier transform of the one-bit wide short scar. The frame expansion of the inverse Fourier transform of the top right part produced a five-bit wide segment. The width of this segment was reduced to one bit shown as a negative in the bottom left part of the figure. A multiple of the bottom left part of the figure, as a positive, was subtracted from the top left part to produce the restored woman figure shown in the bottom right part. In this case, only one frame wavelet was used as highpass filter in the top right part of the figure in the Fourier domain. Using a second filter in the lower left part of the Fourier domain does not seem to modify the final result.

In Fig. 5, the boy image with a diagonal line is restored. One notices in the top right part of the figure the narrow width of the negative of the Fourier transform of the one-bit wide long diagonal line. The frame expansion of the inverse Fourier transform of the top right part produced an eight-bit wide segment. The width of this segment was reduced to one bit. Moreover, fine tuning required that the fourth root of this segment be taken. The result is shown as a negative in the bottom left part of the figure. A multiple of the bottom left part of the figure, as a positive, was subtracted from the top left part to produce the restored woman figure shown in the bottom right part. In this case, two frame wavelets were used as highpass filters in the top right and bottom left parts of the figure in the Fourier domain. Using only one filter in the upper right or lower left part in the Fourier domain does not seem to modify the final result.

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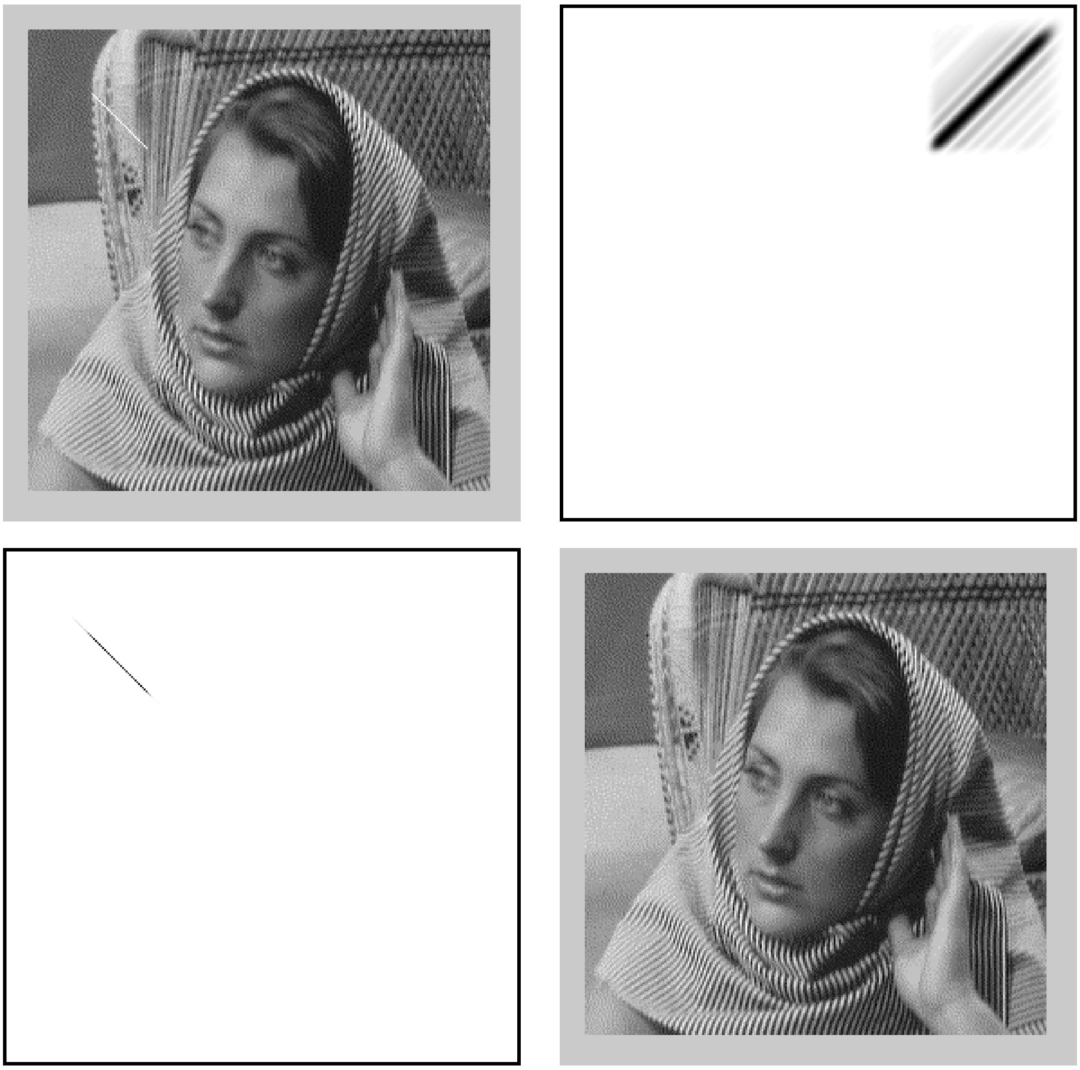

Figure 4: Top left: positive scarred woman figure. Top right: framed negative filtered Fourier transform of top left figure. Bottom left: framed negative frame expansion of the inverse Fourier transform of top right figure. Bottom right: positive restored woman figure.

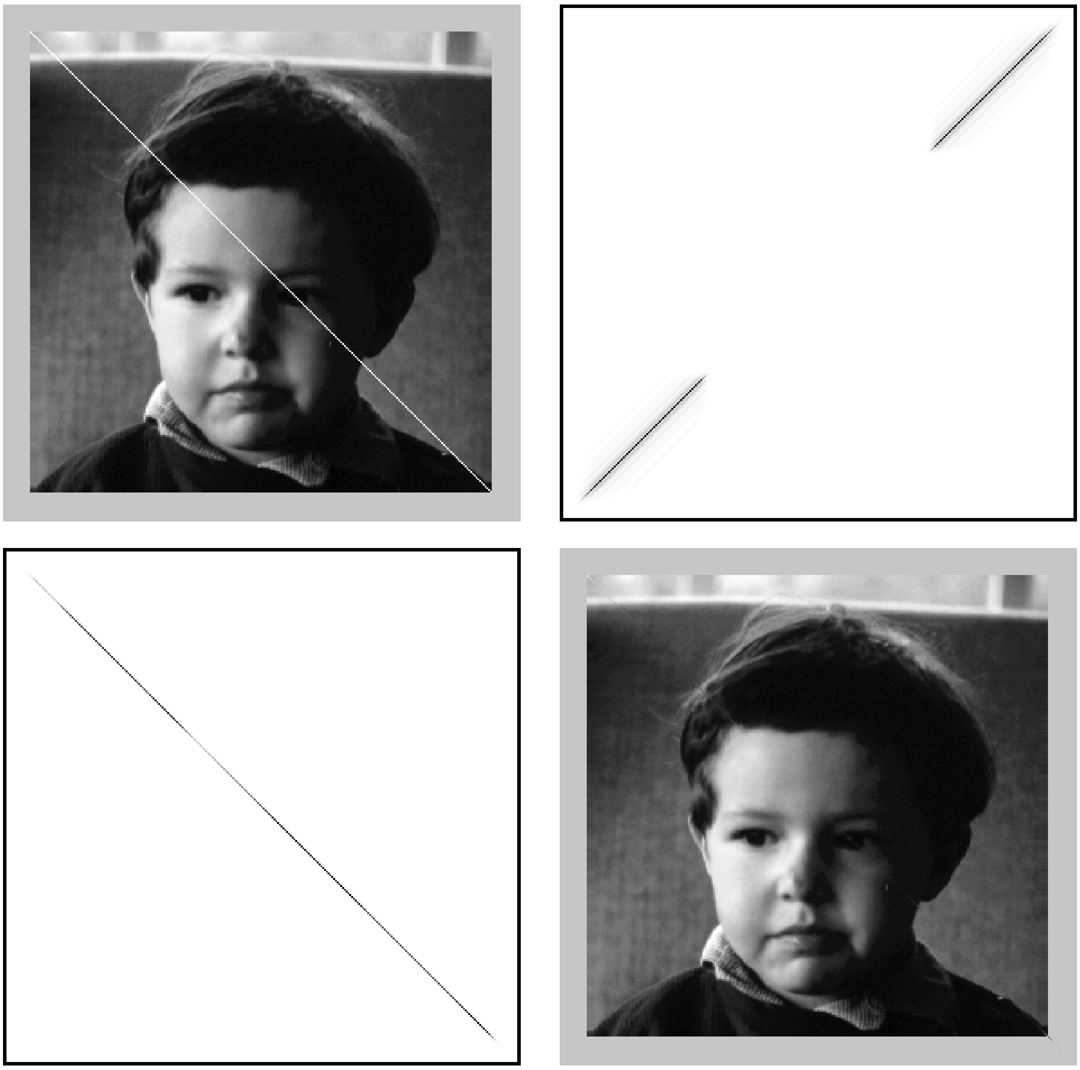

Figure 5: Top left: positive boy figure with diagonal line. Top right: framed negative filtered Fourier transform of top left figure. Bottom left: framed negative frame expansion of the inverse Fourier transform of top right figure. Bottom right: positive restored boy figure.

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