IMAGE DENOISING USING SPLINE AND BLOCK SINGULAR VALUE DECOMPOSITION

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1. Introduction

Image restoration is a central problem in image processing. In practical applications, image denoising is applied to produce good estimates of original images from noisy observations [4]. In recent years, denoising image data has been an active area of research with several different approaches being proposed using techniques such as wavelet thresholding, bilateral filtering and non-linear filtering based on singular value decomposition (SVD) [5].

2. Block singular value denoising

In [2], Devčić and Lončarić presented an algorithm for filtering noise based on nonlinear block processing of images using singular value decomposition (SVD). Noise filtering is performed in the singular value and singular vector domains. A priori knowledge of the noise variance is not required because an estimate of the singular value noise variance is performed in the first phase of the procedure. Filtering is based on eliminating changes in singular values and singular vectors caused by additive Gaussian white noise or other types of noise. Processing the image in smaller blocks makes the SVD procedure computationally feasible.

The general denoising procedure for block singular value decomposition (BSVD) involves four steps: decompose the image into small blocks, factor each block by the singular value decomposition method, eliminate changes in singular values and singular vectors in each block, and reconstruct the image.

Let the original, clean, image, F, be represented as a $K \times L$ matrix. Adding noise to F produces the noised image G of the same size,

$$G = F + N, \tag{1}$$

where N is a random $K \times L$ noise field. Added noise degrades the original information contained in F.

The noised image is divided into square blocks of size $b \times b$. For simplicity, we suppose that $K = kb$ and $L = lb$ (see Fig. 1). Each block has the SVD representation

$$G_{ij} = U_{ij} S_{ij} V_{ij}^T, \qquad i = 1, 2, \dots, k, \quad j = 1, \dots, l, \tag{2}$$

where U_{ij} is the $b \times b$ unitary matrix of left singular vectors, S_{ij} is the $b \times b$ diagonal matrix of singular values, and V_{ij} is the $b \times b$ unitary matrix of right singular vectors [3].

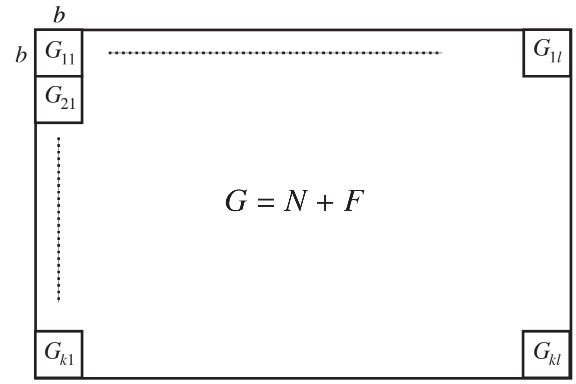

FIGURE 1. Decomposition of noised image G into $b \times b$ blocks.

Equation (2) can be interpreted as an exterior product expansion of the base image,

$$G_{ij} = \sum_{r=1}^{R} U_{ijr} S_{ijr} V_{ijr}^{T}, \tag{3}$$

where R is the rank of G_{ij}, S_{ijr} are the singular values, U_{ijr} are left singular vectors, and V_{ijr} are the right singular vectors. Image degradation, that includes blurring and noising, is reflected in changes in singular values and singular vectors. This is true for the SVD representation of the whole $K \times L$ image, as well as for the SVD representation of the individual $b \times b$ blocks.

The average sum of the last t singular values is calculated over every block:

$$n_s = \frac{1}{k \times l} \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{r=b-t+1}^{b} s_{ijr}. \tag{4}$$

This is not a true value of noise variance, but a value that is proportional to it. Previously calculated SVD of image blocks will now be used for filtering.

The first step in the filtering decreases the noised singular values for every block:

$$\hat{s}_{ijr} = s_{ijr} - p_1 n_s w(r), \tag{5}$$

where $\hat{s}_{ijr}$ is a filtered singular value, p_1 is an image dependent parameter and $w(r)$ is a weighting function that determines the percentage of the estimated noise variance to be subtracted from the noised singular values s_{ijr}. The weighting function used in [2] is

$$w(r) = 1 - \left(1 - \frac{r-1}{b/2}\right)^2, \qquad r = 1, 2, \ldots, b, \tag{6}$$

which is an interpolating parabolic function. A plot of this function for $b = 32$ is shown as the dashed curve in Fig. 5.

In the second step, the singular vectors are slightly filtered to avoid catastrophic changes in images. Singular vectors are first transformed by the discrete Fourier transform (DFT) and then part of the Fourier transform that corresponds to the higher frequencies is reduced by a factor p_2. The filtering operation is performed on both left and right singular vectors.

After filtering has been applied to the three matrices of the SVD of each block of the noised image, the filtered blocks, $\widehat{F}_{ij}$, are calculated:

$$\widehat{F}_{ij} = \widehat{U}_{ij}\widehat{S}_{ij}\widehat{V}ij^T, \qquad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, l, \tag{7}$$

where $\widehat{U}_{ij}$ is the $b \times b$ matrix of filtered left singular vectors, $\widehat{S}_{ij}$ is the $b \times b$ diagonal matrix of restored singular values, and $\widehat{V}_{ij}$ is the $b \times b$ matrix of filtered right singular vectors. The complete filtered image F is reconstructed using the filtered blocks.

3. Objective measures for numerical results

Objective criteria are needed to evaluate the performance of a noise removal scheme. In case of images, the search for simple and suitable criteria is hindered by the fact that the results obtained by statisttical performance criteria may not agree with the subjective evaluation by human eyes.

Given an original image, F, the signal to noise ratio (SNR) in db in the noised image G is expressed in terms of the Frobenius norm, $\|\cdot\|_{\text{fro}}$, by the formula

$$\text{SNR} = 10\log_{10}\left(\frac{\|F\|^2_{\text{fro}}}{\|F-G\|^2_{\text{fro}}}\right), \quad \text{where} \quad \|A\|_{\text{fro}} = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|A_{ij}|^2\right]^{1/2}. \tag{8}$$

Normally, for low-noise images, SNR should be bigger than 20db. We shall compare the restored images, $\hat{F}$, by means of the signal to noise ratio gain (SNRG) in db defined as

$$\text{SNRG} = 10\log_{10}\left(\frac{\|F-G\|^2_{\text{fro}}}{\|F-\widehat{F}\|^2_{\text{fro}}}\right). \tag{9}$$

4. The weighting function problem

In the BSVD method, a weighting function (see formula (5)) is used to estimate the percentage of the noise difference between the singular values of the noised and the original images. This estimate directly influences the accuracy of the BSVD method, but after many experiments, it was found that the weighting function (6) with blocksize 32×32, used in [2] and plotted as the dashed line in Fig. 5 does not sufficiently match the real situations. It is remarked in [2] that in practice, the parabola (6) is often slanted to the left or to the right.

To show this point, we computed the difference between the singular values of noised and original images. We first obtain empirical data for this difference for several images and different noises and plot the curves of the normalized averaged differences as in the following procedure.

(1) Add noise to original images by means of the Matlab `imnoise` function.
(2) Decompose the noised and oringal images into 32×32 blocks and apply the BSVD approach to factor each block into matrices of left and right singular vectors and singular values.
(3) Average the differences of the singular values of the noised image and the original image for all the blocks, normalize the values of the differences between zero and one and plot the resulting curve.

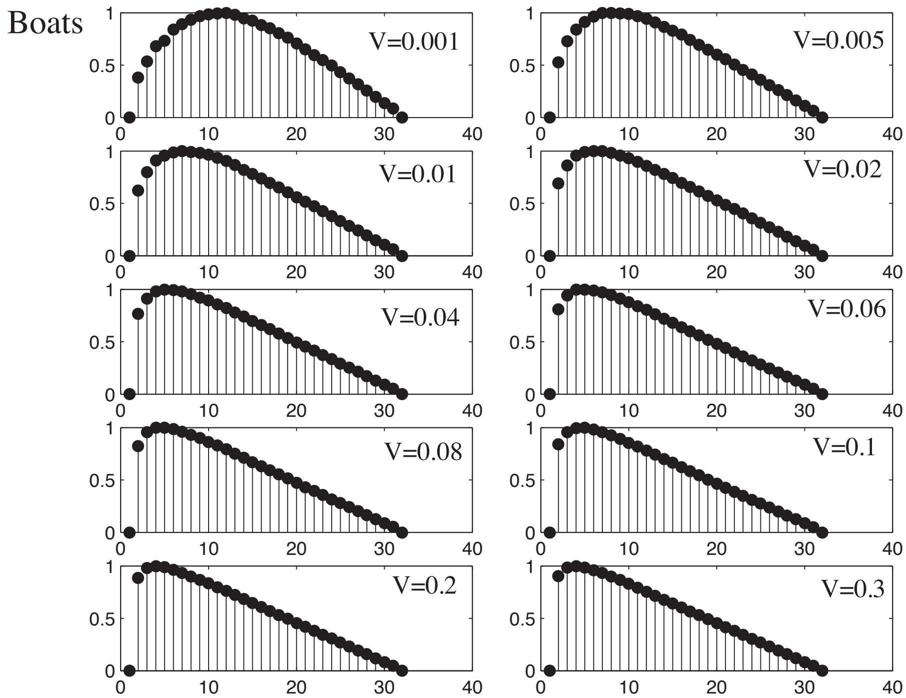

FIGURE 2. Normalized averaged difference in singular values of non-zero Gaussian noised and original Boats image with indicated variance.

In this paper, we report on three sets of experiments. In the first set, zero-mean Gaussian white noised is added to the Boats image with ten values of the variance to obtain curves of the normalized averaged difference between the singular values of the original and the noised images as shown in Fig. 2.

In the second set of curves, the previous procedure is applied to the Lena image with salt and pepper noise of density $0 : 0.1 : 0.9$. The normalized averaged difference between the original and noised singular values is shown in Fig. 3.

In the third set of curves, the same procedure is applied to the Goldhill image with speckle noise with variance $0 : 0.1 : 0.9$. The normalized averaged difference between the original and noised singular values is shown in Fig. 4.

One sees that the above graphs differ from the weighting function (6) shown as the dashed curve in Fig. 5. Several experimental points are listed as follows.

(1) Although the location of the top point C changes slightly in each situation, the common shape and scale of the singular value difference curves is rather stable except in some special cases. It does not change sensibly with several images, noise types, block sizes and noise densities.
(2) The value of the first point of the difference is very unstable and does not characterize the behavior of the whole curve. We ignore this unstable point and set the default value of the first point to 0 and then normalize the values of the curve between 0 and 1.

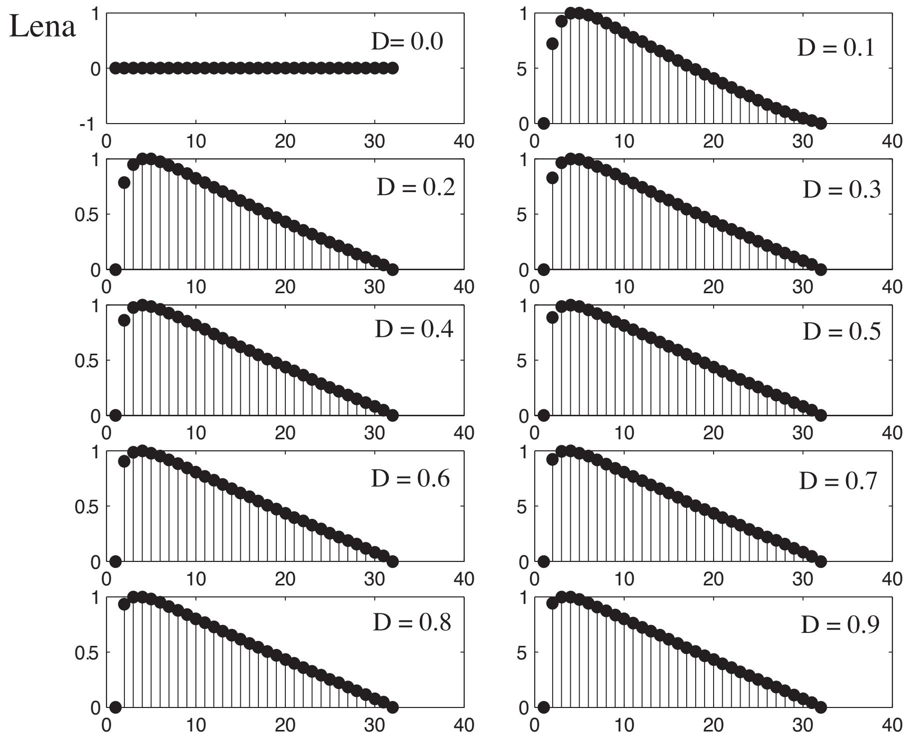

FIGURE 3. Normalized averaged difference in singular values of salt and pepper noised and original Lena image with density $D = 0 : 0.1 : 0.9$.

(3) Several experiments have shown that changing the block size does not influence the shape of the curves. We compared the same noised images with different block sizes from $b = 16, 32, 64, 128$ and the results corroborateed this conjecture.

5. Spline interpolation

The strategy is to find a new weighting function in order to build an estimated singular value difference curve instead of the dashed curve in Fig. 5 which does a poor matching. In Section 4, we have already found the general behavior of the experimental noise difference curves shown in Figs. 2, 3 and 4. They are smooth curves similar to the solid-line curve in Fig. 5.

We want to construct a target curve with top point $C = (c, 1)$ that fits the experimental data (i, y_i), $i = 1, 2, \ldots, 32$, better than the dashed curve (6) with top point C' shown in Fig. 5. Referring to Fig. 6, we first add the left endpoint $A = (0, 0)$ and the right endpoint $E = (33, 0)$ and discard the point $(1, y_1)$ which is an unstable outlier. It will be enough to construct a twice continously differentiable cubic spline made of two cubic polynomials $p_1(x)$ on $[0, c]$ and $p_2(x)$ on $[c, 33]$,

$$p_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \qquad p_2(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3, \tag{10}$$

which interpolate the data $A = (0, 0)$, $C = (c, 1)$ and $E = (33, 0)$ with slopes $k_2 = y_3 - y_2$ at $B = (2, y_2)$ and $k_{32} = y_{31} - y_{32}$ at $D = (32, y_{32})$.

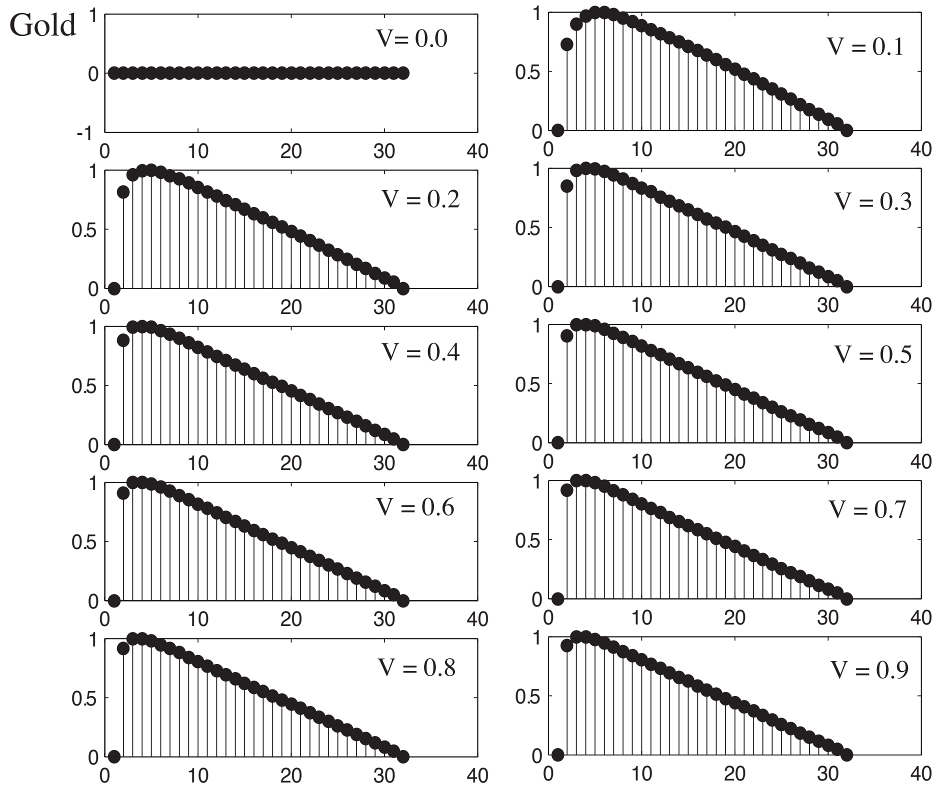

FIGURE 4. Difference in singular values of speckle noised and original Goldhill image with variance $V = 0 : 0.1 : 0.9$.

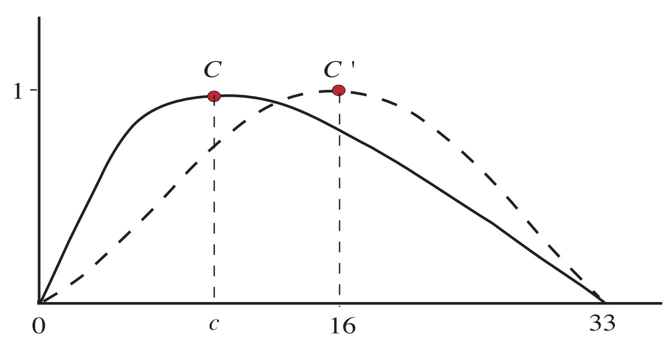

FIGURE 5. Weight function (6) (dashed line) and target (solid line) singular value difference curve.

Thus, the spline equations are:

$$p_1(c) = p_2(c) = 1, \qquad p_1'(c) = p_2'(c) = 0, \qquad p_1''(c) = p_2''(c), \tag{11}$$

$$p_1(0) = 0, \qquad p_2(33) = 0, \tag{12}$$

$$p_1'(2) = k_2 = y_3 - y_2, \qquad p_2'(32) = k_{32} = y_{31} - y_{32}, \tag{13}$$

$$\tag{14}$$

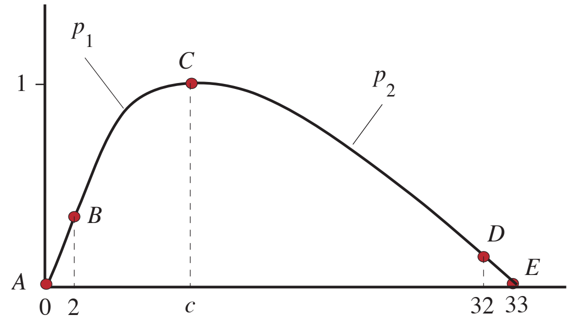

FIGURE 6. Target spline curve.

TABLE 1. Parameter range.

Parameters	Ideal range	Default
c	$[6, 10]$	7
k_2	$[0.28, 0.42]$	0.31
k_3	$[-0.01, -0.06]$	-0.04

which we rewrite in the matrix form $M\mathbf{u} = \mathbf{r}$:

$$\begin{bmatrix} 3c^3 & 2c^2 & c & 1 & -3c^3 & -2c^2 & -c & -1 \\ 3c^2 & 2c & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6c & 2 & 0 & 0 & -6c & -2 & 0 & 0 \\ 3\times 4 & 2\times 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3c^2 & 2c & 1 & 0 \\ 0 & 0 & 0 & 0 & 33^3 & 33^2 & 33 & 1 \\ 0 & 0 & 0 & 0 & 3\times 33^3 & 2\times 33 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_2 \\ 0 \\ 0 \\ k_{32} \end{bmatrix} \qquad (15)$$

Solving this system, we obtain the coefficients a_0, a_1, a_2, a_3, and b_0, b_1, b_2, b_3, of the polynomials $p_1(x)$ and $p_2(x)$, respectively.

We remark that, in the above calculation, k_2, k_{32}, and c are the only free parameters that are used for determining the new weighting function. After many tests we collected lots of experimental data for the range of these parameter. However, in each case, one can try to adjust each value to obtain the best denoising result. In our experiment, we used the default values listed in Table 1.

Table 2 lists the variance and density used to obtain desired SNR levels in noised images. It also lists the SNR gain in db of the BSVD method and the spline method applied to Boats and Lena with zero-mean Gaussian noise, Fp1 and Lena with salt and pepper noise, and Goldhill and Yogi with speckle noise, at six given SNR's in db, respectively. Lastly, the table lists the difference between the SNR gain of the spline method and the SNR gain of the BSVD method,

We remark that with the spline method, any attempt to denoise the singular vectors did not improve the results. Thus the main noise effect lies in the singular values.

Table 2. Differnce in the SNR gain of the spline method and the SNR gain of the BSVD method for a given noised image with listed SNR level.

Zero-mean Gaussian noise added to Boats						
Variance	0.003	0.005	0.0075	0.015	0.05	0.13
SNR of noised Boats (db)	20.0	17.7	15.9	12.9	8.2	5.1
BSVD gain (db)	2.59	3.40	3.85	4.39	4.82	4.89
Spline gain (db)	3.77	4.59	5.17	6.04	6.98	7.28
SNR gain difference (db)	1.18	1.19	1.32	1.65	2.16	2.39
Zero-mean Gaussian noise added to Lena						
Variance	0.0025	0.005	0.0075	0.015	0.05	0.13
SNR of noised Lena (db)	20.3	17.3	15.6	12.6	7.9	4.9
BSVD gain (db)	2.65	3.70	4.14	4.62	4.93	4.96
Spline gain (db)	4.42	5.33	5.83	6.58	7.26	7.58
SNR gain difference (db)	1.77	1.63	1.69	1.96	2.33	2.62
Salt and pepper noise added to Fp1						
Density	0.01	0.1	0.2	0.3	0.4	0.5
SNR of noised Fp1 (db)	22.2	12.2	9.2	7.4	6.2	5.2
BSVD gain (db)	1.39	3.25	4.12	4.25	4.19	4.02
Spline gain (db)	0.44	2.76	4.89	5.43	5.32	4.99
SNR gain difference (db)	-0.95	-0.49	0.77	1.18	1.13	0.97
Salt and pepper noise added to Lena						
Variance	0.01	0.05	0.1	0.2	0.3	0.4
SNR of noised Lena (db)	19.8	12.7	9.7	6.7	5.0	3.7
BSVD gain (db)	1.39	2.74	3.79	4.54	4.63	4.63
Spline gain (db)	0.95	1.84	4.02	6.33	6.82	6.98
SNR gain difference (db)	-0.44	-0.90	0.23	1.79	2.19	2.35
Speckle noise added to Goldhill						
Variance	0.01	0.02	0.05	0.1	0.2	0.4
SNR of noised Gold (db)	20.1	17.2	13.4	10.5	7.7	5.0
BSVD gain (db)	2.61	3.40	4.0	4.23	4.39	4.48
Spline gain (db)	3.18	3.93	4.77	5.25	5.63	5.78
SNR gain difference (db)	0.57	0.53	0.77	1.02	1.24	1.30
Speckle noise added to Yogi						
Variance	0.01	0.02	0.05	0.1	0.3	0.5
SNR of noised Yogi (db)	20.7	17.7	13.8	10.9	6.3	4.4
BSVD gain (db)	-0.73	0.93	2.45	3.21	3.83	4.01
Spline gain (db)	3.16	3.45	3.80	4.24	4.81	5.1
SNR gain difference (db)	3.89	2.52	1.35	1.03	0.98	1.09

6. Conclusion

In this paper experimental weight spline functions have been found which improve on the block singular value decompostion in denoising images. With this spline method it was found that it was better to denoise only the singular values of the singular value decomposition of noised images and not denoise the singular vectors.

7. Acknowledgments

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Qi W., Morimoto A., Ashino R., Vaillancourt R. Image denoising using spline and block singular value decomposition.
A new spline interpolation algorithm is used to denoise images with non-linear filtering based on block singular value decomposition (BSVD). The proposed algorithm was favorably tested under different types of images and a wide range of signal to noise ratios (SNR). Numerical results demonstrate that the new method remove more noise than BSVD.