CURVE FITTING OF IRREGULARLY SAMPLED DATA BY
MULTIWAVELETS NEURAL NETWORKS

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Abstract. Unshifted and shifted multiscaling functions are used as mathematical models for curve fitting of irregularly sampled data. This pre-processing procedure combined with multiwavelet neural networks for data-adaptive curve fitting is shown to perform well in the case of high resolution. In the case of low resolution it is more accurate than numerical integration and cheaper than matrix inversion. In the case of large data, it saves memory as compared with the conjugate gradient method for the same computational cost.

1. Introduction

Scientists and engineers represent empirical data by means of properly chosen mathematical models in order to extract important characteristics from the data, such as the rate of change along a curve, local minima and maxima of a function, the area under a curve, and so on. The goal of curve fitting is to find the parameter values of the chosen mathematical model so that the model fits best the given data. The models to which data are fitted depend on adjustable parameters. To perform curve fitting, one uses a function to measure the closeness between the data and the model. One minimizes this function with respect to the parameters of the model and thus obtains the best-fitting parameters. In many applications, one uses least squares fitting as appropriate measure.

In some cases, the parameters are the coefficients of the terms of the model, for example, if the model is a polynomial or Gaussian distribution. In the simplest case, known as linear regression, a straight line is fitted to the data. In most scientific and engineering models, however, the dependent variable depends on the parameters in a nonlinear way.

Wavelets are a class of functions used to localize a given function in both space and scaling. A family of wavelets can be constructed from a function called wavelet function. Wavelets pick up the details at various scales. Another function, called scaling function, is used to pick up approximations at various scales. The simplest scalar wavelet in $L^2(\mathbb{R})$ is the Haar system, see Meyer [1, Section 3.2], with the indicator function of the interval $[0, 1]$ as scaling function.

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Multiwavelets consist in several scaling functions and wavelets. It is believed that multiwavelets are ideally suited to multichannel signals like color images which are two-dimensional three-channel signals and stereo audio signals which are one-dimensional two-channel signals. For instance, for a two-channel signal, which consists of a two-vector sequence of bits, \( \{x_k\} \), the lowpass and highpass filters are \( 2 \times 2 \) matrix functions corresponding to two scaling functions and two wavelets, respectively. Multiscaling functions and multiwavelets can simultaneously have orthogonality, linear phase, symmetry and compact support. This situation cannot occur in the scalar case with real scaling functions and real wavelets.

In this paper, multiscaling functions for multiwavelets are chosen as the mathematical model for curve fitting. It is known that linear combinations of shifted and dilated multiscaling functions can approximate any function in \( L^2(\mathbb{R}) \). However, before applying the discrete multiwavelet transform to a given data, we need to know the coefficients of the linear combination of shifted and dilated multiscaling functions fitted to the data. The more accurate is this step, the better the discrete multiwavelet transform performs. For this reason, the curve fitting procedure is essential and we call the procedure pre-processing. Neural networks allow data-adaptive curve fitting and adaptability greatly reduces the computation cost for certain problems. In [2], a pre-processing design is proposed for multiwavelet filtering using neural networks for regularly sampled data. In this paper, a curve fitting method for irregularly sampled data is proposed which can be applied to a pre-processing design for the discrete multiwavelet transform.

2. MULTIWAVELETS

Definitions and properties of multiwavelets, filters and filter banks can be found, for instance, in Ashino, Nagase, and Vaillancourt [3] and Zheng [4] and in the monograph by Keinert [5]. The following standard wavelet and multiwavelet notation will be used.

**Notation 1.** The notation is as follows.

(a) Given a function \( f \in L^2(\mathbb{R}) \) and integers \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z} \), we let \( f_{jk}(x) \) denote the scaled and shifted function

\[
 f_{jk}(x) = 2^{j/2} f(2^j x - k).
\]

(b) Given a vector-valued function \( F = [f^1, \ldots, f^d]^T \in L^2(\mathbb{R})^d \), we let \( F_{jk} \) denote the scaled and shifted vector functions

\[
 F_{jk} = [f^1_{jk}, \ldots, f^d_{jk}]^T, \quad j \in \mathbb{Z}, \ k \in \mathbb{Z}.
\]

(c) \( D = \{1, \ldots, d\} \) for a positive integer \( d \).

(d) \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) is the set of natural numbers including zero.

(e) \( \langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x) \, dx \) is the \( L^2(\mathbb{R}) \) inner product of \( f \) and \( g \).

**Definition 1.** A vector-valued function \( \Psi := [\psi^1, \ldots, \psi^d]^T \in L^2(\mathbb{R})^d \) is called a multiwavelet function if the system

\[
 \{\psi_{jk}\}_{\delta \in D, j \in \mathbb{Z}, k \in \mathbb{Z}}
\]
forms an orthonormal basis for $L^2(\mathbb{R})$. In this case, the functions $\psi_{jk}^\delta$ are called multiwavelets and the orthonormal basis $\{\psi_{jk}^\delta\}_{\delta \in D, j, k \in \mathbb{Z}}$ is called an orthonormal multiwavelet basis. The multiwavelet expansion of $f \in L^2(\mathbb{R})$ with respect to an orthonormal multiwavelet basis is

$$f(x) = \sum_{\delta \in D, j, k \in \mathbb{Z}} \langle f, \psi_{jk}^\delta \rangle \psi_{jk}^\delta(x). \quad (3)$$

To construct a multiwavelet function, $\Psi$, from a multiscaling function, $\Phi$, we generalize to multiwavelets the notion of multiresolution analysis given in Mallat [6] and Meyer [1] for scalar wavelets.

**Definition 2.** An increasing sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$,

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots,$$

is called a multiwavelet multiresolution analysis if it satisfies the following four properties:

(a) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\cup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.
(b) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$.
(c) $f(x) \in V_0$ if and only if $f(x - k) \in V_0$ for every $k \in \mathbb{Z}$.
(d) There exists a multiscaling function $\Phi := [\varphi^1, \ldots, \varphi^d]^T \in V_0^d$ such that $\{\varphi(x - k)\}_{\delta \in D, k \in \mathbb{Z}}$ form an orthonormal basis of $V_0$.

When multiwavelets are constructed from a multiresolution analysis, there exist functions $\varphi^\delta$, $\delta \in D$, called scaling functions, such that the set of functions

$$\{\varphi_{0,k}^\delta\}_{\delta \in D, k \in \mathbb{Z}} \cup \{\psi_{jk}^\delta\}_{\delta \in D, j, k \in \mathbb{Z}}$$

is an orthonormal basis of $L^2(\mathbb{R})$. The multiwavelet expansion of $f \in L^2(\mathbb{R})$ with respect to this orthonormal basis is

$$f(x) = \sum_{\delta \in D, j, k \in \mathbb{Z}} \langle f, \varphi_{0,k}^\delta \rangle \varphi_{0,k}^\delta(x) + \sum_{\delta \in D, j, k \in \mathbb{Z}} \langle f, \psi_{jk}^\delta \rangle \psi_{jk}^\delta(x). \quad (4)$$

The coefficients $\langle f, \varphi_{0,k}^\delta \rangle$ and $\langle f, \psi_{jk}^\delta \rangle$ are called multiscaling coefficients and multiwavelet coefficients, respectively.

**Remark 1.** In the $n$-dimensional case, a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ for multiwavelets is defined the same way as in the one-dimensional case, but there are $2^n - 1$ multiwavelet functions which can be parameterized by the set $E := \{0, 1\}^n \setminus \{(0, \ldots, 0)\}$ as

$$\Psi_\varepsilon := [\psi_1^\varepsilon, \ldots, \psi_n^\varepsilon]^T \in V_1^d, \quad \varepsilon \in E.$$

A multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ can be constructed from a given one-dimensional multiresolution analysis by means of the tensor product of multiresolution analyses.

Assume that we have a multiwavelet multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$. Using notation (2), we define the lowpass matrix coefficients

$$H_k := \langle \Phi_{0,0} \Phi_{1,k}^T \rangle_{L^2(\mathbb{R})} = \left[\langle \varphi_{0,0}^\delta, \varphi_{1,k}^\delta \rangle_{L^2(\mathbb{R})}\right]_{(\delta,0) \in D \times D} \in \mathbb{C}^{d \times d},$$
and the matrix frequency response, or matrix symbol,

$$M_0(\xi) := \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} H_k e^{-ik\xi} \in L^2([0, 2\pi])^{d \times d}.$$ 

Then the dilation equation and its Fourier transform are

$$\Phi(x) = 2^{1/2} \sum_{k \in \mathbb{Z}} H_k \Phi(2x - k), \quad \hat{\Phi}(\xi) = M_0(\xi/2) \hat{\Phi}(\xi/2),$$

where \(\hat{\Phi}(\xi) := [\hat{\varphi}^1(\xi), \ldots, \hat{\varphi}^d(\xi)]^T \in L^2(\mathbb{R})^d\). It is known that if we choose \(M_1(\xi)\) such that

$$M(\xi) := \begin{bmatrix} M_0(\xi) & M_0(\xi + \pi) \\ M_1(\xi) & M_1(\xi + \pi) \end{bmatrix}$$

is a unitary matrix for almost all \(\xi \in [0, 2\pi]\), then the multiwavelet function \(\Psi\) is given by the wavelet dilation equation or by its Fourier transform,

$$\Psi(x) = 2^{1/2} \sum_{k \in \mathbb{Z}} G_k \Phi(2x - k), \quad \hat{\Psi}(\xi) = M_1(\xi/2) \hat{\Phi}(\xi/2),$$

where \(G_k, k \in \mathbb{Z}\), are the Fourier coefficients of \(M_1(\xi)\), that is,

$$M_1(\xi) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} G_k e^{-ik\xi} \in L^2([0, 2\pi])^{d \times d}.$$ 

Thus, \(G_k, k \in \mathbb{Z}\), are given by the scalar products

$$G_k := \langle \Psi_{0,0}, \Phi_{1,k}^T \rangle_{L^2(\mathbb{R})} = \left[ \langle \psi^\delta_0, \varphi^\eta_1 \rangle_{L^2(\mathbb{R})} \right]_{(\delta,\eta) \in D \times D} \in \mathbb{C}^{d \times d}.$$

### 3. Curve fitting of irregularly sampled data

The points \(\{x_n\}_{n \in \mathbb{Z}}\) are called the sampling points. For given sampling points \(\{x_n\}_{n \in \mathbb{Z}}\) and a function \(f\), the sequence \(\{f(x_n)\}_{n \in \mathbb{Z}}\) is called the sampled data or the sampling data. For each pair of successive sampling points, \(x_{n-1}\) and \(x_n\), define the sampling width by \(\Delta_n := x_n - x_{n-1}\). If all the sampling widths are equal, the sampling points, \(\{x_n\}_{n \in \mathbb{Z}}\), are said to be regularly sampled, otherwise they are irregularly sampled.

Curve fitting differs from sampling data in some sense. Here, points \(\{(x_n, y_n)\}_{n \in \mathbb{Z}}\) are given in the plane and one tries to fit a curve \(f(x; a_1, a_2, \ldots, a_r)\) to the points in the weighted least squares sense by minimizing the sum of the squares

$$\sum_n |f(x_n; a_1, a_2, \ldots, a_r) - y_n|^2 \left( \frac{\Delta_{n+1} + \Delta_{n-1}}{2} \right)$$

over the parameters \(\{a_j\}\).
3.1. Curve fitting using multiscaling functions. Hereafter, we only deal with the real-valued case and assume that the number of multiscaling functions is two, that is, \( d = 2 \).

First, let us find the best approximation of \( f \in L^2(\mathbb{R}) \) in \( V_j \). As each element \( s_j \in V_j \) is represented as
\[
s_j(x) = \sum_k c_{1,j,k} \phi_1(2^j x - k) + \sum_k c_{2,j,k} \phi_2(2^j x - k),
\]
our problem is to find coefficients \( c_{1,j,k} \) and \( c_{2,j,k} \) that minimize the integral
\[
\int_{\mathbb{R}} |f(x) - s_j(x)|^2 \, dx.
\]
When \( \Phi = [\phi_1, \phi_2]^T \) is an orthonormal multiscaling function, the best approximation is given by
\[
c_{1,j,k} = 2^j \int f(x) \phi_1(2^j x - k) \, dx, \quad c_{2,j,k} = 2^j \int f(x) \phi_2(2^j x - k) \, dx,
\]
which can be calculated by numerical integration.

When an irregularly sampled data \( \{f(x_n)\}_{n \in \mathbb{Z}} \) is given, integral (6) can be approximated by
\[
E(c_{1,j,k}, c_{2,j,k}) = \sum_n |f(x_n) - s_j(x_n)|^2 \left( \frac{x_{n+1} - x_{n-1}}{2} \right).
\]
We expect the least square solution to (8) to be an accurate approximation at the points \( x_n \) and the function \( s_j(x) \) defined by the least square solution gives “the best-fitting” curve to the given irregularly sampled data.

3.2. Curve fitting using shifted multiscaling functions. For certain types of multiscaling functions, \( \Phi \), and sampling points, \( \{x_n\} \), it often happens that a given data \( f(x_n) \) cannot be approximated well by (5). For example, the multiscaling functions \( CL_2 \) and \( CL_3 \), which will be discussed in section 5, have a structural problem in solving a finite dimensional version of the equation
\[
2 \sum_{\ell=1}^2 \sum_{k \in \mathbb{Z}} c_{\ell,j,k} \phi_{\ell,k}(x_n) = f_{j,n},
\]
where \( f_{j,n} \) are determined from \( j \) and \( f(x_n) \). More precisely, when the left-hand side of (9) is represented in matrix form:
\[
A \left[ \ldots, c_{1,j,k}, \ldots, c_{2,j,k}, \ldots \right]^T,
\]
where the components of \( A \) are \( \phi_{\ell,k}(x_n) \), a finite dimensional approximation of \( A \) is a singular matrix. In such a case, we propose to use a shifted function \( s_j(x + \theta) \) instead of \( s_j(x) \), where the shift parameter \( \theta \) will be chosen so as to avoid such a structural problem. We call this procedure a shifted scaling fitting and its algorithm is as follows.
Algorithm 1 (Shifted scaling fitting). Minimize
\[
E_\theta(c_{j,k}^1, c_{j,k}^2) := \sum_n |f(x_n) - s_j(x_n + \theta)|^2 \left( \frac{x_{n+1} - x_{n-1}}{2} \right) \tag{11}
\]
over \( c_{j,k}^1 \) and \( c_{j,k}^2 \) for fixed \( \theta \).

4. MULTIWAVELET NEURAL NETWORKS

The field of neural networks started some fifty years ago but has found solid application only in the past twenty years and it is developing rapidly. Neural networks described in Rumelhart and McClelland [7] are composed of simple elements operating in parallel. These elements are inspired by biological nervous systems. As in nature, the network function is determined largely by the connections between elements.

Assume that each summation of \( \varphi_1(2^j x - k) \) and \( \varphi_2(2^j x - k) \) in (5) contains \( L \) terms and consider a three-layer neural network with input \( x \) and output \( s_j(x) \) as shown in Figure 1. Then, the back-propagation learning method gives the least square solution to (11). We call such a neural network a multiwavelet neural network.

Figure 1. A three-layer multiwavelet neural network.

4.1. Training algorithm. A neural network described in Demuthl and Beale [8] can be trained to perform a particular function by properly choosing the values of the connections (weights) between elements. Commonly, neural networks are adjusted, or trained, so that a particular input leads to a specific target output. The network is adjusted by comparing the output and the target, until the network output matches the target. Typically, many such input/target pairs are used, in this supervised learning, to train a network.

The training algorithm for our multiwavelet neural networks consists in the following four steps.
Algorithm 2 (Training algorithm). Let input $x$ be given.

(i) Fix the resolution $j$ and set $m = 0$, the number of trainings. Let the initial coefficients $c_{j,k}^{\ell}$, $\ell = 1,2$, be properly chosen values. Set the conjugate gradients $dc_{j,k}^{\ell[0]} = 0$. Calculate the initial square error $E^{[0]} = E_\theta \left( c_{j,k}^{1,0}, c_{j,k}^{2,0} \right)$.

(ii) Choose a constant $0 < \lambda^{[m]} < 1$ and calculate the conjugate gradients as follows:

$$dc_{j,k}^{\ell[m+1]} = \frac{\partial E_\theta \left( c_{j,k}^{1,m}, c_{j,k}^{2,m} \right)}{\partial c_{j,k}^{\ell,m}} + \lambda^{[m]} dc_{j,k}^{\ell,m}.$$

(iii) Choose a constant $\eta^{[m]} > 0$ and calculate the new coefficients

$$c_{j,k}^{\ell[m+1]} = c_{j,k}^{\ell,m} - \eta^{[m]} dc_{j,k}^{\ell[m+1]}.$$

(iv) Calculate the square error

$$E^{[m+1]} = E_\theta \left( c_{j,k}^{1,m+1}, c_{j,k}^{2,m+1} \right).$$

If $E^{[m+1]}$ is small enough, then the training is good and the algorithm is stopped. Else if the relative error,

$$\frac{E^{[m]} - E^{[m+1]}}{E^{[m]}},$$

is small, then the algorithm is aborted and we conclude that more training is hopeless and a larger resolution $j$ is needed for this experiment. Otherwise, set $m = m + 1$ and go to (ii).

5. Numerical results

In our numerical experiments, the shifted scaling fitting is applied to various kinds of data. We state here only one of them.

In multiwavelet neural networks, the pairs of input and output $\{x, s_j(x + \theta)\}$ are known and the coefficients $\{c_{j,k}^1, c_{j,k}^2\}$ are unknown. We will deal with overdetermined systems. In this case, the number of input and output pairs $\{x, s_j(x + \theta)\}$ exceeds the number of unknown coefficients $\{c_{j,k}^1, c_{j,k}^2\}$.

5.1. Multiscaling functions used in our numerical experiments. The following three types of multiscaling functions have been used in our numerical experiments.

- **ANV2−3 and ANVb4.4**: The multiscaling functions, with supports $[0, 3]$, $[0, 3]$ and supports $[-3, 3]$, $[-3, 4]$, respectively, of Ashino, Nagase, and Vaillancourt [3] are generated by Daubechies’ compactly supported scalar wavelets with $N = 2$ and the biorthogonal wavelet $9/7$, respectively.

- **CL2 and CL3**: The multiscaling functions of Chui and Lian [9] with $N = 2$ and $N = 3$, respectively, with support $[0, N]$. 

• **GHM**: The multiscaling function of Geronimo, Hardin, and Massopust [10] and its multiwavelet function given in Donavan, Geronimo, Hardin, and Massopust [11]. The supports of the two components of the multiscaling function are $[0, 1]$ and $[0, 2]$, respectively.

5.2. **Data used in the numerical experiments.** To generate an irregularly sampled data, we use the following function $\text{sin245}$:

$$\text{sin245}(x) = \sin \frac{\pi}{20} x + \sin \frac{\pi}{40} x + \sin \frac{\pi}{50} x, \quad 0 \leq x \leq 1000,$$

which is illustrated in Figure 2.

![Figure 2. The graph of the function sin245.](image)

Irregular sampling widths are generated by uniformly random numbers in $[0.1, 1.9]$ and an irregularly sampled data $\text{ir245}$ is given by sampling the function $\text{sin245}$ at the sampling points. A noisy irregularly sampled data $\text{noised-ir245}$ is given by adding white noise ranged in $[-0.2, 0.2]$ to $\text{ir245}$. Figure 3 illustrate these irregularly sampled data, where $x$ denotes their irregular sampling points.

5.3. **Curve fitting of ir245 and noised-ir245 by $V_{-3}$.** Considering the irregular sampling widths of the data, it is necessary that the resolution $j \leq -2$ gives rise to overdetermined systems. Here we choose $j = -3$. We will use the following notation listed in Table 1.

<table>
<thead>
<tr>
<th>$j$</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_c$</td>
<td>Number of coefficients</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Shift parameter</td>
</tr>
<tr>
<td>$N_l$</td>
<td>Number of learnings</td>
</tr>
<tr>
<td>$E_l$</td>
<td>Square error after learning</td>
</tr>
<tr>
<td>$E_M$</td>
<td>Max error after learning</td>
</tr>
</tbody>
</table>
The accuracy of curve fitting of \textit{ir245} by $V_{-3}$ and that of \textit{noised-ir245} by $V_{-3}$ are given in Table 2 and in Table 3, respectively.
Table 3. Accuracy of curve fitting of noised-ir245 by $V_{-3}$.

<table>
<thead>
<tr>
<th>Wavelet</th>
<th>$N_c$</th>
<th>$\theta$</th>
<th>$N_\ell$</th>
<th>$E_\ell$</th>
<th>$E_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GHM</td>
<td>252</td>
<td>0</td>
<td>10</td>
<td>9.80E+00</td>
<td>2.77E-01</td>
</tr>
<tr>
<td></td>
<td>254</td>
<td>0.1</td>
<td>10</td>
<td>9.89E+00</td>
<td>2.83E-01</td>
</tr>
<tr>
<td>CL2</td>
<td>252</td>
<td>0</td>
<td>12</td>
<td>1.03E+01</td>
<td>3.05E-01</td>
</tr>
<tr>
<td></td>
<td>254</td>
<td>0.1</td>
<td>10</td>
<td>1.01E+01</td>
<td>3.05E-01</td>
</tr>
<tr>
<td>CL3</td>
<td>254</td>
<td>0</td>
<td>11</td>
<td>9.50E+00</td>
<td>2.82E-01</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>0.1</td>
<td>11</td>
<td>9.32E+00</td>
<td>2.77E-01</td>
</tr>
<tr>
<td>ANV2−3</td>
<td>254</td>
<td>0</td>
<td>10</td>
<td>1.06E+01</td>
<td>2.79E-01</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>0.1</td>
<td>10</td>
<td>1.01E+01</td>
<td>2.94E-01</td>
</tr>
<tr>
<td></td>
<td>256</td>
<td>0.2</td>
<td>10</td>
<td>1.04E+01</td>
<td>3.13E-01</td>
</tr>
<tr>
<td>ANVb4.4</td>
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<td>9.33E+00</td>
<td>2.79E-01</td>
</tr>
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<td>0.1</td>
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<td>9.69E+00</td>
<td>2.80E-01</td>
</tr>
</tbody>
</table>

Computation was done with MATLAB version 6.5.1 on a PC running under Windows 2000 with 512MB of RAM. The CPU is AMD Athlon 1.13GHz.

6. Conclusion

Various numerical experiments have led us to the following conclusion.

- In the case of high resolution, the accuracy of our curve fitting method greatly surpasses the accuracy of the method of numerical integration.
- In the case of low resolution, our curve fitting method is one-digit more accurate than the method of numerical integration and it is cheaper than the method of inversion by means of the MATLAB \ operator.
- Our curve fitting method for irregularly sampled data can be applied to a pre-processing design for the discrete multiwavelet transform.

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