RESTORATION OF LOST SAMPLES BY OVERSAMPLING NEAR THE NYQUIST RATE *

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Abstract. A formula in matrix form for restoration is given and the matrix with the sampling rate near the Nyquist rate is investigated. Elements of this matrix can be expanded into the Laurent series of the sampling rate parameter, which is defined by the quotient of the Nyquist rate and the sampling rate. The Nyquist rate corresponds to a pole. First terms of these Laurent series near the Nyquist rate are given.

Key words. oversampling, restoration, lost sample, Laurant series.

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1. Introduction. Let us start with well-known Shannon's sampling theorem:

THEOREM 1.1. *Assume that* $f \in L^2(\mathbf{R})$ *and* $\operatorname{supp} \widehat{f} \subset [-\sigma, \sigma]$, *where* $\sigma > 0$. *Put* $t_k := k\pi/\sigma$. *Then*

$$f(t) = \sum_{k \in \mathbf{Z}} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)}, \tag{1.1}$$

where the convergence is absolutely uniform on every compact set [1].

Here, $\widehat{f}$ denotes the Fourier transform of f, which and whose inverse $\mathcal{F}^{-1}$ are defined by

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-it\cdot\omega} f(t)\, dt, \qquad (\mathcal{F}^{-1}g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{it\cdot\omega} g(\omega)\, d\omega.$$

The points $\{t_k\}_{k\in\mathbf{Z}}$ are called the *sampling points* and the series $\{f(t_k)\}_{k\in\mathbf{Z}}$ are called the *sampling data*. The function $f \in L^2(\mathbf{R})$ is said to be *band-limited* to $[-\sigma, \sigma]$, if $\operatorname{supp} \widehat{f} \subset [-\sigma, \sigma]$. The sampling frequency σ/π in (1.1) is known as the *Nyquist rate*, which is the minimum rate at which the band-limited function to $[-\sigma, \sigma]$ needs to be sampled in order to be reconstructed perfectly from the sampling data.

Fix δ with $\delta > \sigma$. Then $\operatorname{supp} \widehat{f} \subset [-\delta, \delta]$ so that Theorem 1.1 implies

$$f(t) = \sum_{k \in \mathbf{Z}} f(\eta_k) \frac{\sin \delta(t - \eta_k)}{\delta(t - \eta_k)} \tag{1.2}$$

with the sampling points $\{\eta_k\}_{k\in\mathbf{Z}}$ defined by $\eta_k := k\pi/\delta$. The sampling frequency δ/π in (1.2) is called the *sampling rate*.

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[1] Here and in the sequel $\left.\frac{\sin t}{t}\right|_{t=0} = 1$ by definition.

Denote by $\chi_{(-\sigma,\sigma)}(\omega)$ the characteristic function of the interval $(-\sigma,\sigma)$ and put

$$g_\sigma(t) := \sqrt{\frac{2}{\pi}}\frac{\sin\sigma t}{t} = \mathcal{F}^{-1}\chi_{(-\sigma,\sigma)}. \tag{1.3}$$

Then we have

$$g_\delta * g_\sigma = g_\sigma, \tag{1.4}$$

since

$$\widehat{g_\delta * g_\sigma} = \widehat{g}_\delta * \widehat{g}_\sigma = \chi_{(-\delta,\delta)}\chi_{(-\sigma,\sigma)} = \chi_{(-\sigma,\sigma)} = \widehat{g}_\sigma.$$

Noting $\widehat{f}(\omega) = \widehat{f}(\omega)\chi_{(-\sigma,\sigma)}(\omega)$ and (1.4), we have

$$\begin{aligned} f(t) &= (f * g_\sigma)(t) \\ &= \sum_{k\in\mathbf{Z}} f(\eta_k)\Big(\frac{\sin\delta(t-\eta_k)}{\delta(t-\eta_k)} * g_\sigma\Big) \\ &= \delta^{-1}\sqrt{\frac{\pi}{2}}\sum_{k\in\mathbf{Z}} f(\eta_k)\big(g_\delta * g_\sigma\big)(t-\eta_k) \\ &= \delta^{-1}\sum_{k\in\mathbf{Z}} f(\eta_k)\frac{\sin\sigma(t-\eta_k)}{(t-\eta_k)}. \end{aligned}$$

Define $r := \sigma/\delta (0 < r < 1)$, which is called the *sampling rate parameter*. Then the above formula reduces to

$$\begin{aligned} f(t) &= r\sum_{k\in\mathbf{Z}} f(\eta_k)\frac{\sin\sigma(t-\eta_k)}{\sigma(t-\eta_k)} \\ &= r\sum_{k\in\mathbf{Z}} f(\eta_k)\frac{\sin(\sigma t - rk\pi)}{\sigma t - rk\pi}. \end{aligned} \tag{1.5}$$

Let n be a positive integer and $\mathcal{M} = \{m_0, \ldots, m_{n-1}\} \subset \mathbf{Z}$ be the index set of lost samples. For $m \in \mathcal{M}$, substitute $t = \eta_m$ in (1.5). Then

$$f(\eta_m) = r\Big\{\sum_{\ell\in\mathcal{M}} f(\eta_\ell)\frac{\sin r\pi(m-\ell)}{r\pi(m-\ell)} + \sum_{k\in\mathbf{Z}\setminus\mathcal{M}} f(\eta_k)\frac{\sin r\pi(m-k)}{r\pi(m-k)}\Big\},$$

and so

$$\begin{aligned} &\sum_{\ell\in\mathcal{M}}\Big\{\delta_{m,\ell} - r\frac{\sin r\pi(m-\ell)}{r\pi(m-\ell)}\Big\}f(\eta_\ell) \\ &\qquad = r\sum_{k\in\mathbf{Z}\setminus\mathcal{M}}\frac{\sin r\pi(m-k)}{r\pi(m-k)}f(\eta_k), \qquad m\in\mathcal{M}, \end{aligned} \tag{1.6}$$

where $\delta_{m,\ell}$ is the Kronecker's delta. The cardinal sine function is defined by

$$\operatorname{sinc} t := \begin{cases} \dfrac{\sin\pi t}{\pi t} & (t \neq 0), \\ 1 & (t = 1), \end{cases}$$

which is an entire function. Denote $s_k(t) := \operatorname{sinc} kt$ and define the matrices

$$S_{\mathcal{M}}(r) := \big(s_{m-\ell}(r)\big)_{(m,\ell)\in\mathcal{M}\times\mathcal{M}},$$
$$S_{\mathbf{Z}\setminus\mathcal{M}}(r) := \big(s_{m-k}(r)\big)_{(m,k)\in\mathcal{M}\times(\mathbf{Z}\setminus\mathcal{M})}.$$

Then (1.6) is represented as

$$\big(I - rS_{\mathcal{M}}(r)\big)\big(f(\eta_\ell)\big)_{\ell\in\mathcal{M}} = rS_{\mathbf{Z}\setminus\mathcal{M}}(r)\big(f(\eta_k)\big)_{k\in\mathbf{Z}\setminus\mathcal{M}}, \tag{1.7}$$

where $\big(f(\eta_\ell)\big)_{\ell\in\mathcal{M}}$ is the column vector of the lost samples and $\big(f(\eta_k)\big)_{k\in\mathbf{Z}\setminus\mathcal{M}}$ is the column vector of the known samples. Thus we can regain the lost samples and $f(t)$ itself from the remaining samples, if $I - rS_{\mathcal{M}}(r)$ is invertible. This formula (1.7) was given first in Robert J. Marks II [4] [2] and can be found in various books, for example, Robert J. Marks II [5] or Ahmed I. Zayed [8]. In [4], it seems that the author knows that the matrix $\big(I - rS_M(r)\big)$ is invertible for $0 < r < 1$, but there is no proof. One of our purpose in this paper is to give a proof of this fact. That is to show

Claim 1: *The matrix $I - rS_{\mathcal{M}}(r)$ is invertible for $0 < r < 1$,*
whose proof will be given in Chapter 2.

From the engineering point of view, as $r = 1$ corresponds to the Nyquist rate, it is desired to analyse $(I - rS_M)^{-1}$ near $r = 1$, so we put $x = 1 - r$. Then $0 < x < 1$ and $r = 1$ corresponds to $x = 0$. Since $\sin \pi(m-\ell)r = (-1)^{m-\ell+1} \sin \pi(m-\ell)x$, we have

$$\begin{aligned} I - rS_{\mathcal{M}} &= \big(\delta_{m,\ell} - rs_{m-\ell}(r)\big)_{(m,\ell)\in\mathcal{M}\times\mathcal{M}} \\ &= \big((-1)^{m-\ell} x s_{m-\ell}(x)\big)_{(m,\ell)\in\mathcal{M}\times\mathcal{M}} \end{aligned}$$

and

$$\begin{aligned} rS_{\mathbf{Z}\setminus\mathcal{M}} &= \big(rs_{m-k}(r)\big)_{(m,k)\in\mathcal{M}\times(\mathbf{Z}\setminus\mathcal{M})} \\ &= \big((-1)^{m-k+1} x s_{m-k}(x)\big)_{(m,k)\in\mathcal{M}\times(\mathbf{Z}\setminus\mathcal{M})}. \end{aligned}$$

We denote

$$T_{\mathcal{M}} := \big(I - rS_{\mathcal{M}}\big)/x = \big((-1)^{m-\ell} s_{m-\ell}(x)\big)_{(m,\ell)\in\mathcal{M}\times\mathcal{M}} \tag{1.8}$$

and

$$T_{\mathbf{Z}\setminus\mathcal{M}} := \big(rS_{\mathbf{Z}\setminus\mathcal{M}}\big)/x = \big((-1)^{m-k+1} s_{m-k}(x)\big)_{(m,k)\in\mathcal{M}\times(\mathbf{Z}\setminus\mathcal{M})}. \tag{1.9}$$

Then, dividing (1.7) by x and using the invertibility of $T_{\mathcal{M}}$ for $0 < x < 1$ by Claim 1, we can restore lost samples $\big\{f(\eta_\ell)\big\}_{\ell\in\mathcal{M}}$ from known samples $\big\{f(\eta_k)\big\}_{k\in\mathbf{Z}\setminus\mathcal{M}}$ by the following new formula:

[2] The restoration formula given in Theorem [4, p.754] used the invertibility of the matrix $(\exp(-\sqrt{-1}\pi m u_p/B))|_{(m,p)\in\mathcal{M}\times\mathcal{M}}$ for any $u_p (p \in \mathcal{M})$ which are nonequal but otherwise arbitrary chosen from the interval $W < u < B$, where W and B are given positive constants. But this is not the case. For example, let $\mathcal{M} = \{0, m_1\}$ and $m_1$be so large that $(m_1 - 3)/m_1 > W/B$, and put $u_0 = [(m_1 - 1)/m_1]B, u_1 = [(m_1 - 3)/m_1]B$. Then this matrix is not invertible.

New formula

$$\big(f(\eta_\ell)\big)_{\ell\in\mathcal{M}} = T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}} \big(f(\eta_k)\big)_{k\in\mathbf{Z}\setminus\mathcal{M}}. \tag{1.10}$$

As the elements of the matrix $T_{\mathcal{M}}$ are entire functions of x, the elements of $T_{\mathcal{M}}^{-1}$ have the Laurent series in general. More precisely, we can show

Claim 2: *The elements of $T_{\mathcal{M}}^{-1}$ have the Laurent series near $x = 0$, whose leading terms can be given explicitly.*

When we caluculate the right hand side of (1.10) in the order

$$T_{\mathcal{M}}^{-1}\Big(T_{\mathbf{Z}\setminus\mathcal{M}} \big(f(\eta_k)\big)_{k\in\mathbf{Z}\setminus\mathcal{M}}\Big),$$

Claim 2 (*i.e.* Theorem 4.1) gives us an approximation of (1.10) near the Nyquist rate.

On the contrary, when we caluculate the right hand side of (1.10) in the order

$$\Big(T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}\Big) \big(f(\eta_k)\big)_{k\in\mathbf{Z}\setminus\mathcal{M}}, \tag{1.11}$$

we need the calculation of $\Big(T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}\Big)$ near $x = 0$. As for this, we can show

Claim 3: *The elements of $T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}$ have the power series near $x = 0$ (i.e. they have no singularity at $x = 0$), whose leading terms can be given explicitly.*

One may thought that Claim 3 would give us another approximation formula of (1.10). But unfortunately, (1.11) with $\Big(T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}\Big)$ replaced by its leading term does not have meaning for general $\big(f(\eta_k)\big)_{k\in\mathbf{Z}\setminus\mathcal{M}}$.

The proofs of Claim 2 and 3 will be given in Chapter 4. Chapter 3 is devoted to prepare linear algebraic lemmas. Lemma 3.2 is new and very useful.

2. The invertibility of the matrix $I - rS_{\mathcal{M}}(r)$.

Theorem 2.1. *The matrix $I - rS_{\mathcal{M}}$ is positive definite so that it is invertible.*

Proof. It suffices to show

$$(rS_{\mathcal{M}}(r)\xi, \xi) < (\xi, \xi), \tag{2.1}$$

for any non-trivial $\xi = (\xi_m)_{m\in\mathcal{M}}$, where $(\ ,\ )$ denotes the standard inner product.

Noting that

$$r \operatorname{sinc} rt = \frac{1}{2\pi}\int_{-\pi r}^{\pi r} e^{ity}\,dy,$$

which follows from (1.3) with $\sigma = \pi r$, and that

$$rS_{\mathcal{M}}(r)|_{r=1} = I : \text{the identity matrix},$$

we have

$$\begin{aligned}
(rS_{\mathcal{M}}(r)\xi, \xi) &= \sum_{m,\ell\in\mathcal{M}} \frac{1}{2\pi}\int_{-\pi r}^{\pi r} e^{i(m-\ell)y}\xi_m\xi_\ell\,dy \\
&= \frac{1}{2\pi}\int_{-\pi r}^{\pi r} |\sum_{m\in\mathcal{M}} e^{imy}\xi_m|^2\,dy \\
&< \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sum_{m\in\mathcal{M}} e^{imy}\xi_m|^2\,dy \\
&= (rS_{\mathcal{M}}(r)\xi, \xi)|_{r=1} = (\xi, \xi),
\end{aligned}$$

which shows (2.1). ∎

Let $\mu_j(x)$ be the eigenvalues of $T_{\mathcal{M}}$ ordered as $\mu_0 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > 0$. Using the above formula and the min-max principle, we can show easily that

$$\mu_j(x) \leq \text{ Const. } x^{2j}.$$

And more careful calculation shows

$$\mu_j(x) = \mu_j^0 x^{2j}(1 + O(x^2)) \quad \mu_j^0 > 0,$$

which will be shown in [1].

3. Linear algebraic lemmas. In this section, we shall give some linear algebraic lemmas needed for calculating leading terms of the Laurent series of the elements of the inverse matrix $T_{\mathcal{M}}^{-1}$.

For $\ell = (\ell_0, \ldots, \ell_{n-1}) \in \mathbf{Z}^n$, we put $|\ell| := \sum \ell_{q=0}^{n-1} \ell_q$. We put $L_n := \{0, \ldots, n-1\}$ and denote the permutation group on $L_n = \{0, \ldots, n-1\}$ by S_n.

Lemma 3.1. *Let $\lambda \in \mathbf{C}\backslash\{0\}$. Then, for a matrix $\big(a_{pq}\big)_{(p,q)\in L_n\times L_n}$,*

$$\det\big(\lambda^{\ell_p+m_q} a_{pq}\big)_{(p,q)\in L_n\times L_n} = \lambda^{|\ell|+|m|} \det\big(a_{pq}\big)_{(p,q)\in L_n\times L_n}. \tag{3.1}$$

Proof. Since

$$\big(\lambda^{\ell_p+m_q} a_{pq}\big)_{(p,q)\in L_n\times L_n}$$

$$= \begin{pmatrix} \lambda^{\ell_0} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\ell_{n-1}} \end{pmatrix} \big(a_{pq}\big)_{(p,q)\in L_n\times L_n} \begin{pmatrix} \lambda^{m_0} & & 0 \\ & \ddots & \\ 0 & & \lambda^{m_{n-1}} \end{pmatrix},$$

the identity (3.1) is obvious. ∎

We denote the difference product by

$$\Delta(\ell) := \prod_{i>j} (\ell_i - \ell_j).$$

For any function f and for any $x, y \in \mathbf{C}$, we put

$$A_f(x, y; \ell, m) := \det\big(f(\ell_p x - m_q y)\big)_{(p,q)\in L_n\times L_n}.$$

Then we have

Lemma 3.2. *Let f be a holomorphic function defined in a neighbourhood of the origin with the Taylor expansion*

$$f(t) = \sum_{j=0}^{\infty} \frac{a_j}{j!} t^j. \tag{3.2}$$

Then we have

$$A_f(x, y; \ell, m) = \Delta(\ell)\Delta(m) x^{n(n-1)/2} y^{n(n-1)/2} \big\{c_n(f) + O\big(\sqrt{x^2 + y^2}\big)\big\} \tag{3.3}$$

as (x,y) *tends to* $(0,0)$. *Here*

$$c_n(f) := \frac{(-1)^{n(n-1)/2}}{\{0!\cdots(n-1)!\}^2}\det\bigl(a_{p+q}\bigr)_{(p,q)\in L_n\times L_n}. \tag{3.4}$$

Proof. By the multi-linear property of the determinant, we have

$$\begin{aligned}
&A_f(x,y;\ell,m) = \det\Bigl(\sum_{j=0}^{\infty}\frac{a_j}{j!}\,(\ell_p x - m_q y)^j\Bigr)_{(p,q)\in L_n\times L_n}\\
&= \sum_{0\le j_0,\dots,j_{n-1}}\frac{a_{j_0}}{j_0!}\cdots\frac{a_{j_{n-1}}}{j_{n-1}!}\det\bigl((\ell_p x - m_q y)^{j_q}\bigr)_{(p,q)\in L_n\times L_n}\\
&= \sum_{0\le j_0,\dots,j_{n-1}}\frac{a_{j_0}}{j_0!}\cdots\frac{a_{j_{n-1}}}{j_{n-1}!}\sum_{0\le t_0\le j_0,\dots,0\le t_{n-1}\le j_{n-1}}{}_{j_0}C_{t_0}\cdots{}_{j_{n-1}}C_{t_{n-1}}\\
&\qquad\times\det\bigl((\ell_p x)^{j_q-t_q}(-m_q y)^{t_q}\bigr)_{(p,q)\in L_n\times L_n}\\
&= \sum_{0\le j_0,\dots,j_{n-1}}\frac{a_{j_0}}{j_0!}\cdots\frac{a_{j_{n-1}}}{j_{n-1}!}\sum_{0\le t_0\le j_0,\dots,0\le t_{n-1}\le j_{n-1}}{}_{j_0}C_{t_0}\cdots{}_{j_{n-1}}C_{t_{n-1}}\\
&\qquad\times(-m_0)^{t_0}\cdots(-m_{n-1})^{t_{n-1}}y^{|t|}\det\bigl((\ell_p x)^{s_q}\bigr)_{(p,q)\in L_n\times L_n},
\end{aligned}$$

where we put $s_i = j_i - t_i (i = 0,\cdots,n-1)$ and $|s| := \sum_{i=0}^{n-1} s_i$.

For $\ell = (\ell_0,\dots,\ell_{n-1})$ and $s = (s_0,\dots,s_{n-1})$, we put

$$D(\ell;s) := \det\bigl((\ell_p)^{s_q}\bigr)_{(p,q)\in L_n\times L_n}.$$

Then we have

$$\begin{aligned}
&A_f(x,y;\ell,m)\\
&\quad= \sum_{0\le j_0,\dots,j_{n-1}}\frac{a_{j_0}}{j_0!}\cdots\frac{a_{j_{n-1}}}{j_{n-1}!}\sum_{0\le t_0\le j_0\dots0\le t_{n-1}\le j_{n-1}}{}_{j_0}C_{t_0}\cdots{}_{j_{n-1}}C_{t_{n-1}}\\
&\qquad\qquad\times(-m_0)^{t_0}\cdots(-m_{n-1})^{t_{n-1}}y^{|t|}x^{|s|}D(\ell;s).
\end{aligned}$$

If two of $\ell_0,\dots,\ell_{n-1}$ coincide with each other, $D(\ell;s)=0$. Hence $A_f(x,y;\ell,m)$ can be divided by $\Delta(\ell)$. If two of $s_0,\dots,s_{n-1}$ coincide with each other, $D(\ell;s)=0$. So, non-trivial terms of $A_f(x,y;\ell,m)$ are restricted to the case that $s_0,\dots,s_{n-1}$ are all distinct. The smallest exponent $|s|$ to x occurs only when the mapping $i\mapsto s_i$, which will be denoted by σ, is a permutation of L_n and then $|s| = \sum_{i=0}^{n-1} s_i = n(n-1)/2$. Since $A_f(x,y;\ell,m)$ is symmetric with respect to $\ell_p x$ and $m_q y$, $A_f(x,y;\ell,m)$ can be divided by $\Delta(m)$, non-trivial terms of $A_f(x,y;\ell,m)$ are restricted to the case that $t_0,\dots,t_{n-1}$ are all distinct, and the smallest exponent $|t|$ to y occurs when the mapping $i\mapsto t_i$, which will be denoted by τ, is a permutation of L_n.

Thus the leading term of $A_f(x,y;\ell,m)$ is of order $x^{n(n-1)/2}y^{n(n-1)/2}$ and its coefficeint is given by

$$\begin{aligned}
&\sum_{\sigma,\tau\in S_n}\frac{a_{\sigma(0)+\tau(0)}}{(\sigma(0)+\tau(0))!}\cdots\frac{a_{\sigma(n-1)+\tau(n-1)}}{(\sigma(n-1)+\tau(n-1))!}\\
&\qquad\times{}_{\sigma(0)+\tau(0)}C_{\sigma(0)}\cdots{}_{\sigma(n-1)+\tau(n-1)}C_{\sigma(n-1)}\\
&\qquad\times(-m_0)^{\sigma(0)}\cdots(-m_{n-1})^{\sigma(n-1)}D(\ell;\vec{\tau}),
\end{aligned}\tag{3.5}$$

where $\vec{\tau} := (\tau(0), \dots, \tau(n-1))$. Since $D(\ell; \vec{\tau}) = \operatorname{sgn}\tau\, \Delta(\ell)$, the quantity (3.5) is equal to

$$\begin{aligned}\Delta(\ell) &\sum_{\sigma\in S_n} (-m_0)^{\sigma(0)} \cdots (-m_{n-1})^{\sigma(n-1)} \\ &\times \Big\{ \sum_{\tau\in S_n} \operatorname{sgn}\tau \frac{a_{\sigma(0)+\tau(0)}}{(\sigma(0)+\tau(0))!} \cdots \frac{a_{\sigma(n-1)+\tau(n-1)}}{(\sigma(n-1)+\tau(n-1))!} \\ &\qquad \times {}_{\sigma(0)+\tau(0)}C_{\sigma(0)} \cdots {}_{\sigma(n-1)+\tau(n-1)}C_{\sigma(n-1)} \Big\}.\end{aligned} \tag{3.6}$$

Substitute $\tau = \tau'\sigma$ in (3.6). Then the above quantity is equal to

$$\begin{aligned}&\Delta(\ell) \sum_{\sigma\in S_n} (-m_0)^{\sigma(0)} \cdots (-m_{n-1})^{\sigma(n-1)} \\ &\quad\times\Big\{ \sum_{\tau'\in S_n} \operatorname{sgn}\tau' \operatorname{sgn}\sigma \frac{a_{\sigma(0)+\tau'\sigma(0)}}{(\sigma(0)+\tau'\sigma(0))!} \cdots \frac{a_{\sigma(n-1)+\tau'\sigma(n-1)}}{(\sigma(n-1)+\tau'\sigma(n-1))!} \\ &\qquad \times {}_{\sigma(0)+\tau'\sigma(0)}C_{\sigma(0)} \cdots {}_{\sigma(n-1)+\tau'\sigma(n-1)}C_{\sigma(n-1)} \Big\} \\ &= \Delta(\ell) \sum_{\sigma\in S_n} \operatorname{sgn}\sigma (-m_0)^{\sigma(0)} \cdots (-m_{n-1})^{\sigma(n-1)} \\ &\quad\times\Big\{ \sum_{\tau'\in S_n} \operatorname{sgn}\tau' \frac{a_{\sigma(0)+\tau'\sigma(0)}}{(\sigma(0)+\tau'\sigma(0))!} \cdots \frac{a_{\sigma(n-1)+\tau'\sigma(n-1)}}{(\sigma(n-1)+\tau'\sigma(n-1))!} \\ &\qquad \times {}_{\sigma(0)+\tau'\sigma(0)}C_{\sigma(0)} \cdots {}_{\sigma(n-1)+\tau'\sigma(n-1)}C_{\sigma(n-1)} \Big\}.\end{aligned} \tag{3.7}$$

Since

$$\prod_{i=0}^{n-1} G(\sigma(i)) = \prod_{i=0}^{n-1} G(i)$$

for any G, the quantity (3.7) is equal to

$$\begin{aligned}&\Delta(\ell) \sum_{\sigma\in S_n} \operatorname{sgn}\sigma\, (-m_0)^{\sigma(0)} \cdots (-m_{n-1})^{\sigma(n-1)} \\ &\quad\times \Big\{ \sum_{\tau'\in S_n} \operatorname{sgn}\tau' \frac{a_{0+\tau'(0)}}{(0+\tau'(0))!} \cdots \frac{a_{(n-1)+\tau'(n-1)}}{((n-1)+\tau'(n-1))!} \\ &\qquad \times \frac{(0+\tau'(0))!}{0!\cdot\tau'(0)!} \cdots \frac{((n-1)+\tau'(n-1))!}{(n-1)!\cdot\tau'(n-1)!} \Big\} \\ &= \frac{(-1)^{n(n-1)/2}}{\{0!\cdots(n-1)!\}^2} \Delta(\ell)\Delta(m) \sum_{\tau'\in S_n} \operatorname{sgn}\tau'\, a_{0+\tau'(0)} \cdots a_{(n-1)+\tau'(n-1)} \\ &= \Delta(\ell)\Delta(m) c_n(f).\end{aligned}$$

□

Note that only $\{a_j\}_{j=0,1,\dots,2(n-1)}$ is needed to determine $c_n(f)$.

COROLLARY 3.3. *For* $\lambda \in \mathbf{C}\backslash\{0\}$,

$$c_n\big(f(\lambda t)\big) = \lambda^{n(n-1)} c_n\big(f(t)\big). \tag{3.8}$$

Proof. Applying Lemma 3.1 with $\ell = m = (0, \dots, n-1)$, we have

$$\begin{aligned}c_n\big(f(\lambda t)\big) &= \frac{(-1)^{n(n-1)/2}}{\{0!\cdots(n-1)!\}^2} \det\big(\lambda^{p+q} a_{p+q}\big)_{(p,q)\in L_n\times L_n} \\ &= \frac{(-1)^{n(n-1)/2}}{\{0!\cdots(n-1)!\}^2} \lambda^{n(n-1)/2+n(n-1)/2} \det\big(a_{p+q}\big)_{(p,q)\in L_n\times L_n} \\ &= \lambda^{n(n-1)} c_n\big(f(t)\big).\end{aligned}$$

□

For a real number t, we denote by $[t]$ the largest integer no greater than t.

LEMMA 3.4. *Assume that f is an even holomorphic function in a neighbourhood of the origin having the repersentation* (3.2). *Then*

$$\det\big(a_{p+q}\big)_{(p,q)\in L_n\times L_n} = \det\big(a_{2(p'+q')}\big)_{(p',q')\in L_{[(n+1)/2]}\times L_{[(n+1)/2]}}$$

$$\times \det\big(a_{2(p''+q''+1)}\big)_{(p'',q'')\in L_{[n/2]}\times L_{[n/2]}}.$$

Proof. Since f is even, $a_k = 0$ for any odd k and non-trivial terms of the determinant

$$\det\big(a_{p+q}\big)_{(p,q)\in L_n\times L_n} = \sum_{\tau\in S_n} \operatorname{sgn}\tau\, a_{0+\tau(0)}\cdots a_{(n-1)+\tau(n-1)} \tag{3.9}$$

satisfy $k \equiv \tau(k) \bmod 2$ for $k \in L_n$. Hence every permutation τ giving a non-trivial term of the right hand side of (3.9) can be represented as $\tau = \tau_1\tau_2$, where τ_1 fixes odd numbers and τ_2 fixes even numbers, so that τ_1 can be identified with $\tau' \in S_{[(n+1)/2]}$ by the isomorphism

$$\tau_1(2k') = 2\tau'(k'), \quad k' \in L_{[(n+1)/2]}$$

and τ_2 can be identified with $\tau'' \in S_{[n/2]}$ by the isomorphism

$$\tau_2(2k''+1) = 2\tau''(k'')+1, \quad k'' \in L_{[n/2]}.$$

Since

$$\tau_1\tau_2(2k') = \tau_1(2k') = 2\tau'(k'), \quad k' \in L_{[(n+1)/2]},$$

$$\tau_1\tau_2(2k''+1) = \tau_1(2\tau''(k'')+1) = 2\tau''(k'')+1, \quad k'' \in L_{[n/2]}$$

and

$$\operatorname{sgn}\tau_1 = \operatorname{sgn}\tau', \quad \operatorname{sgn}\tau_2 = \operatorname{sgn}\tau'',$$

the non-trivial terms of the right hand side of (3.9) is

$$\begin{aligned}
&\sum_{\tau=\tau_1\tau_2} \operatorname{sgn}\tau_1\tau_2\, a_{0+\tau_1\tau_2(0)}\cdots a_{2[(n+1)/2]+\tau_1\tau_2(2[(n+1)/2])}\\
&\qquad\times a_{1+\tau_1\tau_2(1)}\cdots a_{2[n/2]+1+\tau_1\tau_2(2[n/2]+1)}\\
&= \sum_{\substack{\tau'\in S_{[(n+1)/2]}\\ \tau''\in S_{[n/2]}}} \operatorname{sgn}\tau'\, a_{2\cdot 0+2\tau'(0)}\cdots a_{2[(n+1)/2]+2\tau'([(n+1)/2])}\\
&\qquad\times \operatorname{sgn}\tau''\, a_{2\cdot 0+1+2\tau''(0)+1}\cdots a_{2[n/2]+1+2\tau''([n/2])+1}\\
&= \det\big(a_{2(p'+q')}\big)_{(p',q')\in L_{[(n+1)/2]}\times L_{[(n+1)/2]}}\\
&\qquad\times \det\big(a_{2(p''+q''+1)}\big)_{(p'',q'')\in L_{[n/2]}\times L_{[n/2]}}.
\end{aligned}$$

□

Now, let us calculate $c_n(f)$ with $f = \operatorname{sinc} t$, which will be denoted by $c_n(\operatorname{sinc} t)$. To do this, we will use the following lemma, which is well-known as the determinant of the Cauchy matrix; see *e.g.* [7, Lemma 7.6.A].

LEMMA 3.5. *Let* $\xi = (\xi_0, \xi_1, \cdots, \xi_{N-1}), \eta = (\eta_0, \eta_1, \cdots, \eta_{N-1}) \in C^N$ *with* $\xi_p + \eta_q \neq 0$ *for any* $p, q \in L_N$. *Then it holds that*

$$\det\Big(\frac{1}{\xi_p+\eta_q}\Big)_{(p,q)\in L_N\times L_N} = \frac{\Delta(\xi)\Delta(\eta)}{\prod_{p,q\in L_N}(\xi_p+\eta_q)}. \tag{3.10}$$

As a corollary, we have

COROLLARY 3.6. *Let* r *be a positive number. Then*

$$\det\Big(\frac{1}{2p+2q+r}\Big)_{(p,q)\in L_N\times L_N} = \frac{\{2!!\cdots(2N-2)!!\}^2}{\prod_{p,q\in L_N}(2p+2q+r)}. \tag{3.11}$$

Proof. Substituting $\xi_p = \eta_p = 2p + r/2$ in (3.10), we have (3.11). ☐

LEMMA 3.7.

$$c_n(\operatorname{sinc} t) = \frac{\pi^{n(n-1)}}{\{3!!\,5!!\cdots(2n-3)!!\}^2(2n-1)!!}. \tag{3.12}$$

Proof. When $f(t) = \operatorname{sinc}(t/\pi)$,

$$a_{2k} = (-1)^k/(2k+1)\,; \qquad a_{2k+1} = 0, \qquad k \in \mathbf{N}\cup\{0\}.$$

By virtue of Corollary 3.3 with $\lambda = 1/\pi$ and Lemma 3.2, we have

$$\begin{aligned} c_n(\operatorname{sinc} t) &= \pi^{n(n-1)} c_n(\operatorname{sinc} t/\pi) \\ &= \frac{(-1)^{n(n-1)/2}\pi^{n(n-1)}}{\{0!\cdots(n-1)!\}^2}\det\big(a_{p+q}\big)_{(p,q)\in L_n\times L_n}. \end{aligned}$$

Since $\operatorname{sinc}(t/\pi)$ is even, Lemma 3.4 and Lemma 3.1 with $\lambda = -1$ yield

$$\begin{aligned} \det\big(a_{p+q}\big)_{(p,q)\in L_n\times L_n} &= \det\Big(\frac{(-1)^{p'+q'}}{2(p'+q')+1}\Big)_{(p',q')\in L_{[(n+1)/2]}\times L_{[(n+1)/2]}} \\ &\quad\times \det\Big(\frac{(-1)^{p''+q''+1}}{2(p''+q''+1)+1}\Big)_{(p'',q'')\in L_{[n/2]}\times L_{[n/2]}} \\ &= \det\Big(\frac{1}{2(p'+q')+1}\Big)_{(p',q')\in L_{[(n+1)/2]}\times L_{[(n+1)/2]}} \\ &\quad\times(-1)^{[n/2]}\det\Big(\frac{1}{2(p''+q''+1)+1}\Big)_{(p'',q'')\in L_{[n/2]}\times L_{[n/2]}}. \end{aligned}$$

Thus we have

$$\begin{aligned} c_n(\operatorname{sinc} t) = &\ \frac{\pi^{n(n-1)}}{\{0!\cdots(n-1)!\}^2} \\ &\times\det\Big(\frac{1}{2p'+2q'+1}\Big)_{(p',q')\in L_{[(n+1)/2]}\times L_{[(n+1)/2]}} \\ &\times\det\Big(\frac{1}{2p''+2q''+3}\Big)_{(p'',q'')\in L_{[n/2]}\times L_{[n/2]}}, \end{aligned} \tag{3.13}$$

where we have used the fact that $n(n-1)/2+[n/2]\equiv 0 \bmod 2$.

Now, let us calculate the each factor of the right hand side of the equation (3.10).

$$\begin{aligned}0!1!\cdots(n-1)! &= 2!\cdots(2[(n+1)/2]-2)!\times 1!3!\cdots(2[n/2]-1)!\\ &= 2!!\cdots(2[(n+1)/2]-2)!!\cdot 3!!\cdots(2[(n+1)/2]-3)!!\\ &\quad\times 2!!\cdots(2[n/2]-2)!!\cdot 3!!\cdots(2[n/2]-1)!!\,,\end{aligned}$$

$$\det\Big(\frac{1}{2p+2q+1}\Big)_{(p,q)\in L_{[(n+1)/2]}\times L_{[(n+1)/2]}} = \frac{\{2!!\cdots(2[(n+1)/2]-2)!!\}^2}{\prod_{p,q\in L_{[(n+1)/2]}}(2p+2q+1)},$$

and

$$\det\Big(\frac{1}{2p+2q+3}\Big)_{(p,q)\in L_{[n/2]}\times L_{[n/2]}} = \frac{\{2!!\cdots(2[n/2]-2)!!\}^2}{\prod_{p,q\in L_{[n/2]}}(2p+2q+3)},$$

where we have used Corollary 3.6 with $r=1, N=[(n+1)/2]$ and with $r=3, N=[n/2]$.

Substituting these equations into (3.13), we have

$$\begin{aligned}c_n = \;& \pi^{n(n-1)}\\ &\times\frac{1}{\{3!!\cdots(2[(n+1)/2]-3)!!\}^2\prod_{0\le p,q\le[(n+1)/2]-1}(2p+2q+1)}\\ &\times\frac{1}{\{3!!\cdots(2[n/2]-1)!!\}^2\prod_{0\le p,q\le[n/2]-1}(2p+2q+3)}.\end{aligned}\tag{3.14}$$

The denominator in the second factor of the equation (3.14) is

$$\prod_{j=1}^{[(n+1)/2]-2}(2j+1)!!\cdot\prod_{j=0}^{[(n+1)/2]-1}\cdot(2j-1)!!\cdot\prod_{0\le p,q\le[(n+1)/2]-1}(2p+2q+1)$$

$$\begin{aligned}&= \prod_{j=1}^{[(n+1)/2]-2}(2j+1)!!\cdot\prod_{p=0}^{[(n+1)/2]-1}\Big\{(2p-1)!!\prod_{q=0}^{[(n+1)/2]-1}(2p+2q+1)\Big\}\\ &= \prod_{j=1}^{[(n+1)/2]-2}(2j+1)!!\cdot\prod_{p=0}^{[(n+1)/2]-1}(2p+2[(n+1)/2]-1)!!\\ &= \prod_{j=1}^{2[(n+1)/2]-2}(2j+1)!!\,.\end{aligned}$$

Similarly, the denominator in the third factor of the equation (3.14) is

$$\prod_{j=1}^{2[n/2]-1}(2j+1)!!\,.$$

Thus we have (3.12). ∎

The following constant d_n will be needed in Section 4.

$$d_n := \begin{cases} 1 & (n=1)\,,\\ \dfrac{c_{n-1}(\operatorname{sinc} t)}{c_n(\operatorname{sinc} t)} & (n\ge 2).\end{cases}\tag{3.15}$$

We have easily

Corollary 3.8.

$$d_n := (2n-3)!!(2n-1)!!\pi^{-2(n-1)}.\tag{3.16}$$

4. Expansion of $T_{\mathcal{M}}^{-1}$ and $T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}$. In this chapter, we shall show Claim 2 as Theorem 4.1 and Claim 3 as Theorem 4.2. Remember that $\mathcal{M} = \{m_0, \ldots, m_{n-1}\}$ with $m_0 < m_1 < \cdots < m_{n-1}$ is the index set of the lost samples. We denote the relative distance between m_p and m_q by

$$m_{pq} := |m_q - m_p| \qquad \text{for} \qquad p, q \in L_n$$

and the product of the inverse of the relative distances with the origin m_q by

$$K_q := \prod_{p \in L_n \setminus \{q\}} \frac{1}{m_{pq}}.$$

Then, we have the following:

THEOREM 4.1. *Let $T_{\mathcal{M}}$ be defined by* (1.8). *Then we have*

$$T_{\mathcal{M}}^{-1} = x^{-2(n-1)} d_n \Big((-1)^{m_p + p} K_p \cdot (-1)^{m_q + q} K_q + O(x^2) \Big)_{(p,q) \in L_n \times L_n} \tag{4.1}$$

as x tends to 0, *where d_n is defined by* (3.15).

Proof. Since

$$T_{\mathcal{M}} = \big((-1)^{m_p - m_q} \operatorname{sinc}(m_p x - m_q x) \big)_{(p,q) \in L_n \times L_n},$$

by Lemma 3.2, we have

$$\begin{aligned} \det T_{\mathcal{M}} &= A_{\operatorname{sinc} t}(x, x; m, m) \\ &= \Delta(m)^2 x^{n(n-1)} \big\{ c_n(\operatorname{sinc} t) + O(x) \big\}, \end{aligned} \tag{4.2}$$

where $m = (m_0, \ldots, m_{n-1})$. We denote

$$m^{(p)} := (m_0, \ldots, m_{p-1}, \widehat{m_p}, m_{p+1}, \ldots, m_{n-1}) \in \mathbf{Z}^{n-1}, \quad p \in L_n.$$

By easy calculation, we have

$$\Delta(m^{(p)}) = \Delta(m) K_p.$$

Applying Lemma 3.1 with $\lambda = -1$ and Lemma 3.2, we have that the cofactors of $T_{\mathcal{M}}$ are

$$\begin{aligned} \Delta_{pq} &= (-1)^{p+q} (-1)^{|m| - m_p} (-1)^{|m| - m_q} A_{\operatorname{sinc} t}(x, x; m^{(p)}, m^{(q)}) \\ &= (-1)^{p+q+m_p+m_q} \Delta(m^{(p)}) \Delta(m^{(q)}) x^{(n-1)(n-2)} \big\{ c_{n-1}(\operatorname{sinc} t) + O(x) \big\} \\ &= (-1)^{p+q+m_p+m_q} \Delta(m)^2 K_p K_q x^{(n-1)(n-2)} \big\{ c_{n-1}(\operatorname{sinc} t) + O(x) \big\}. \end{aligned}$$

Hence,

$$\begin{aligned} &T_{\mathcal{M}}^{-1} \\ &= \Big(\frac{(-1)^{p+q+m_p+m_q} \Delta(m)^2 K_p K_q x^{(n-1)(n-2)} \big\{ c_{n-1}(\operatorname{sinc} t) + O(x) \big\}}{\Delta(m)^2 x^{n(n-1)} \big\{ c_n(\operatorname{sinc} t) + O(x) \big\}} \Big)_{(p,q) \in L_n \times L_n} \\ &\qquad = x^{-2(n-1)} d_n \Big((-1)^{m_p + p} K_p \cdot (-1)^{m_q + q} K_q + O(x) \Big)_{(p,q) \in L_n \times L_n}. \end{aligned}$$

Since $\operatorname{sinc} t$ is even, its Taylor expansion has only even power terms and the remainder term $O(x)$ in $T_{\mathcal{M}}^{-1}$ can be replaced by $O(x^2)$. □

Remark 1: When the distances between the elements of $\mathcal{M}$ go larger, K_p's go smaller and the singularity of $T_{\mathcal{M}}^{-1}$ become smaller. For example, if one element of $\mathcal{M}$ has a distance from the others of order $1/x$, K_p are of order x^{n-1} and the singularity of the leading term of $T_{\mathcal{M}}^{-1}$ disappears.

Now, let us calculate $T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}$.

THEOREM 4.2. *Let $j \in L_n$ and $k \in \mathbf{Z}\setminus\mathcal{M}$. The (j,k) component of $T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}$ is*

$$(-1)^{m_j-k+1} \prod_{p \in L_n \setminus \{j\}} \frac{m_p - k}{m_p - m_j} + O(x^2). \tag{4.3}$$

Proof. Let $m(j,k) \in \mathbf{Z}^n$ be m with the jth component replaced by k. We put the (j,k) component of $T_{\mathcal{M}}^{-1} T_{\mathbf{Z}\setminus\mathcal{M}}$, $C_{j,k}/\det T_{\mathcal{M}}$. By Cramer's formula, we have

$$\begin{aligned}
& C_{j,k} \\
= \;& \text{determinant of the matrix } \Big[((-1)^{m_p-m_q} \operatorname{sinc}(m_p x - m_q x))_{(p,q)\in L_n \times L_n} \\
& \qquad \text{with the } j\text{th column repalced by } (-1)^{m_p-k+1} \operatorname{sinc}(m_p x - kx)\Big] \\
= \;& (-1)^{m_j-k+1} \times \text{determinant of the matrix } \Big[(\operatorname{sinc}(m_p x - m_q x))_{(p,q)\in L_n \times L_n} \\
& \qquad \text{with the } j\text{th column repalced by } \operatorname{sinc}(m_p x - kx)] \\
= \;& (-1)^{m_j-k+1} A_{\operatorname{sinc} t}(x, x; m, m(j,k)) \\
= \;& (-1)^{m_j-k+1} \Delta(m) \Delta(m(j,k)) x^{n(n-1)} \big\{c_n(\operatorname{sinc} t) + O(x)\big\},
\end{aligned}$$

which with (4.2) shows (4.3). □

Remark 2: The leading term of (4.3) is of order $O(k^{n-1})$ as k tends to ∞ so that the determinant defined by its leading term does not give a bounded operator from (ℓ^2) to $\mathbf{C}^n$.

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