Heat extensions, atomic decompositions and Sobolev embeddings on manifolds with bounded geometry

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Let X be an d–dimensional connected Riemannian manifold with bounded geometry. On such Riemannian manifolds the whole scales of Besov $B^s_{p,q}(X)$ and Triebel-Lizorkin $F^s_{p,q}(X)$ spaces were defined via localization principle by H.Triebel in 1986. The second scale covers Sobolev spaces $H^s_p(X) = F^s_{p,2}(X)$, L_p spaces $L_p(X) = F^0_{p,2}(X)$, $1 < p < \infty$ and local Hardy spaces $h_p(X) = F^0_{p,2}(X)$. The fractional Sobolev space $H^s_p(X)$ can be defined in terms of Laplace-Beltrami operator Δ. The Besov scale includes Hölder-Zygmund spaces $\mathcal{C}^s(X) = B^s_{\infty,\infty}(X)$, $s > 0$. An embedding of one such space into another of the form

$$F^{s_0}_{p_0,q_0}(X) \hookrightarrow F^{s_1}_{p_1,q_1}(X)\,, \qquad B^{s_0}_{p_0,q_0}(X) \hookrightarrow B^{s_1}_{p_1,q_1}(X)$$

with $s_0 - \frac{d}{p_0} \geq s_1 - \frac{d}{p_1}$ are called the Sobolev embeddings.

The purpose of our lecture is twofold. First we give "global" characterization of the Besov and Triebel-Lizorkin spaces in terms the heat semi-group related to the Laplace-Beltrami operator Δ. Second we study compactness of Sobolev embeddings of the spaces in presence of symmetries. The both subject are linked by an atomic decomposition of the above function spaces with optimal coefficients. Such the decomposition can be constructed using the heat kernel. It is equivalent to the heat characterization and is a suitable tool of studying properties of the Sobolev embeddings.

The compactness of Sobolev embeddings in presence of symmetries was noticed in late 70-ties by people working in PDE. It can be used in variational methods of solving nonlinear semi-elliptic equations. We study the subject for the subspaces consisted of distributions invariant with respect to a compact group of isometries. We described the sufficient and necessary condition of compactness of the embeddings. Moreover we try to characterize the degree of compactness in terms of entropy and approximation numbers. We described the asymptotic behavior of the entropy and approximation numbers of the Sobolev embeddings of spaces of radial distributions defined on euclidean and hyperbolic spaces. This estimates are of some interest for spectral theory of differential and pseudo-differential operators due to the Carl inequality. Some applications will be described.