GLOBAL BOUNDEDNESS THEOREMS FOR FOURIER INTEGRAL OPERATORS ASSOCIATED WITH CANONICAL TRANSFORMATIONS

MITSURU SUGIMOTO

This talk is based on the joint work with Michael Ruzhansky (Imperial College).

⊙ Fourier integral operators

$$Tu(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y)\, dyd\xi \tag{1}$$

($x \in \mathbf{R}^n$), where $a(x,y,\xi)$ is an amplitude function and $\phi(x,y,\xi)$ is a real phase function of the form

$$\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi).$$

Note that, by the equivalence of phase function theorem, Fourier integral operators with the local graph condition can always be written in this form locally.

Local L^2 mapping property of (1) has been established by Hörmander (1971) and Eskin (1970). The aim of this talk is to establish the global L^2-boundedness properties of operators (1).

⊙ When is T globally L^2-bounded?

The following result of Asada-Fujiwara was motivated by the construction of fundamental solution of Schrödinger equation in the way of Feynman's path integral, and it requires the boundedness of all the derivatives of entries of the matrix

$$D(\phi) = \begin{pmatrix} \partial_x\partial_y\phi & \partial_x\partial_\xi\phi \\ \partial_\xi\partial_y\phi & \partial_\xi\partial_\xi\phi \end{pmatrix}.$$

• (Asada-Fujiwara 1978) Assume that all the derivatives of $a(x,y,\xi)$ and all the derivatives of each entry of $D(\phi)$ are bounded. Also assume that $|\det D(\phi)| \geq C > 0$. Then T is $L^2(\mathbf{R}^n)$-bounded.

However, there one had to make a quite restrictive and not always natural assumption on the boundedness of $\partial_\xi\partial_\xi\phi$, which fails in many important cases.

The case we have in mind is

$$\phi(x,y,\xi) = x\cdot\xi - y\cdot\psi(\xi), \tag{2}$$

where $\psi(\xi)$ is a smooth function of growth order 1. If we take $\psi(\xi) = \xi$, then we have $\phi(x,y,\xi) = x\cdot\xi - y\cdot\xi$, and the operator T defined by it is a pseudo-differential operator.

We cannot use Asada-Fujiwara's result with our example (2), because the boundedness of the entries of $\partial_\xi\partial_\xi\phi$ fails generally.

⊙ Why is the phase function (2) important?

Because it is used to represent a canonical transforms. In fact, if we take $a(x, y, \xi) = 1$, we have

$$Tu(x) = F^{-1}[(Fu)(\psi(\xi))](x) \tag{3}$$

hence

$$T \cdot \sigma(D) = (\sigma \circ \psi)(D) \cdot T.$$

Especially, for a positive and homogeneous function $p(\xi)$ of degree 1, we have the relation

$$T \cdot (-\triangle) = p(D)^2 \cdot T, \tag{4}$$

if we can take

$$\psi(\xi) = p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|}. \tag{5}$$

The L^2-property of the Laplacian $-\triangle$ is well known in various situations. Our objective is to know the L^2-property of the operator T, so that we can extract the L^2-property of the operator $p(D)^2$ from that of the Laplacian.

⊙ Main results

The following is our main result, which is expected to have many applications.

For $m \in \mathbf{R}$, we set

$$\langle x \rangle^m = \left(1 + |x|^2\right)^{m/2}.$$

Let $L^2_m(\mathbf{R}^n)$ be the set of functions f such that the norm

$$\|f\|_{L^2_m(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} |\langle x \rangle^m f(x)|^2 \, dx \right)^{1/2}$$

is finite.

• <u>Theorem 1</u>. Let $\phi(x, y, \xi) = x \cdot \xi - \varphi(y, \xi)$. Assume that

$$|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0,$$

and all the derivatives of entries of $\partial_y \partial_\xi \varphi$ are bounded. Also assume that

$$\left|\partial_\xi^\alpha \varphi(y, \xi)\right| \leq C_\alpha \langle y \rangle \quad \text{for all } |\alpha| \geq 1,$$
$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)\right| \leq C_{\alpha\beta\gamma} \langle x \rangle^{-|\alpha|}$$

for all α, β, and γ. Then T is bounded on $L^2_m(\mathbf{R}^n)$ for any $m \in \mathbf{R}$.

Theorem 1 says that, if amplitude functions $a(x, y, \xi)$ have some decaying properties with respect to x, we do not need the boundedness of $\partial_\xi \partial_\xi \phi$ for the L^2-boundedness, as required in Asada-Fujiwara, and we can have weighted estimates, as well.

The same is true when both phase and amplitude functions have some decaying properties with respect to y.

$\odot$ An example of how to use our results.

The global smoothing property of generalized Schrödinger equations

$$\begin{cases} (i\partial_t - p(D)^2)u(t,x) = 0, \\ u(0,x) = f(x). \end{cases} \tag{6}$$

Kato-Yajima (1989) showed that the classical Schrödinger equation ($p(D)^2 = -\triangle$) has the global smoothing estimate

$$\|\sigma(X,D)u\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \le C\|f\|_{L^2(\mathbf{R}^n_x)}, \tag{7}$$

where $n \ge 3$ and

$$\sigma(X,D) = \langle x \rangle^{-1} \langle D \rangle^{1/2}.$$

From this fact, we can extract a similar estimate for generalized Schrödinger equation (6).

• Assumption. $p(\xi) \in C^\infty(\mathbf{R}^n \setminus 0)$ is homogeneous of order 1, $p(\xi) > 0$, and the hypersurface $\Sigma = \{\xi; p(\xi) = 1\}$ has non-vanishing Gaussian curvature.

The curvature condition on Σ means that the Gauss map

$$\frac{\nabla p}{|\nabla p|} : \Sigma \to S^{n-1}$$

is a global diffeomorphism and its Jacobian never vanishes. Hence, we can construct the inverse C^∞-map $\psi^{-1}(\xi)$ of $\psi(\xi)$ given by (5). Let T be the operator with the phase function $\phi(x,y,\xi) = x \cdot \xi - y \cdot \psi(\xi)$ and the amplitude function $a(x,y,\xi) = 1$. By (3), the inverse T^{-1} can given by replacing ψ by ψ^{-1}.

Then we have the relation

$$T^{-1} \cdot p(D)^2 = (-\triangle) \cdot T^{-1}$$

by (4). Operating T^{-1} from the left hand side of equation (6), we have, by this relation,

$$\begin{cases} (i\partial_t - \triangle)T^{-1}u(t,x) = 0, \\ T^{-1}u(0,x) = T^{-1}f(x). \end{cases}$$

Hence, from (7) and Theorem 1, we obtain the same global smoothing property of (6) as that of classical Schrödinger equation.

• Theorem 2. Suppose $n \ge 3$. Under the assumption above, the solution $u(t,x)$ to equation (6) has estimate (7).

• Remark. Walther (2002) consider the case of radially symmetric $p(\xi)^2$. Theorem 5 says that we can treat more general case.