Asymptotic behaviour of eigenvalues and L^2 normalized eigenfunctions of the Laplace-Beltrami operators on closed Riemannian manifolds

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Let $(M,\ g)$ be a smooth closed Riemannian manifold of dimension $n \geq 2$ and Δ be the positive Laplace-Beltrami operator on M. Let $L_2(M)$ be the space of square integrable functions on M with respect to the Riemannian density $\mathrm{d}x$ associated with the metric g. Let $e_1(x), e_2(x), \cdots$ be a complete real orthonormal basis in $L_2(M)$ for the eigenfunctions of Δ such that $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots$ for the corresponding eigenvalues, where $e_j(x)$ $(j = 1, 2, \cdots)$ are real valued smooth function on M and λ_j are nonnegative real numers. Also, let $\mathbf{e}_j$ denote the projection onto the 1-dimensional space $\mathbf{C}e_j$.

Let $\lambda \in [1, \infty)$. Let $N(\lambda)$ be the number of eigenvalues $\leq \lambda^2$, where we count the multiplicities of eigenvalues. We define the spectral function $e(x,y,\lambda)$ and the unit spectral projection operator χ_λ as follows:

$$e(x,y,\lambda) := \sum_{\lambda_j \leq \lambda} e_j(x)e_j(y),\ \chi_\lambda f := \sum_{\lambda_j \in (\lambda, \lambda+1]} \mathbf{e}_j(f).$$

There are obvious relations among $e(x,y,\lambda)$, $N(\lambda)$ and χ_λ,

$$\int_M e(x,x,\lambda)\,\mathrm{d}x = N(\lambda),\ \chi_\lambda f(x) = \int_M f(y)\big(e(x,y,\lambda+1) - e(x,y,\lambda)\big)\,\mathrm{d}x.$$

Let $p \in [2, \infty]$. We shall review the following topics.

- One term asymptotic formula of $e(x,x,\lambda)$ and $N(\lambda)$ as $\lambda \to \infty$ by Levitan and Avakumovič
- The sharp asymptotic estimates of (L_2, L_p) mapping norms of χ_λ by Sogge
- The Hörmander multiplier theorem on M by Seeger-Sogge

Reset $p \in [2, \infty)$. Let $s \in [0, \infty)$, $r \in (0, \infty)$ and k be a nonnegative integer. Let $H_p^s(M)$ be the fractional Sobolev L_p spaces, $\mathscr{C}^r(M)$ be the Hölder-Zygmund spaces on M. We shall introduce some recent results.

- One term asymptotic formulae of $\partial_x^\alpha \partial_y^\beta e(x, y, \lambda)|_{x=y}$ in a geodesic normal chart on M
- The sharp asymptotic estimates of (L^2, C^k) mapping norms of χ_λ
- The asymptotic estimates of $(L^2, \mathscr{C}^r)$ mapping norms of χ_λ
- The sharp asymptotic estimates of (L^2, H_p^s) mapping norms of χ_λ
- The asymptotic estimates of C^k, $\mathscr{C}^r$ and H_p^s norms of eigenfunctions $e_j(x)$ as $j \to \infty$

We shall also sketch the proof for all the results mentioned as above. The main tools of the proof are the Hadamard parametrix for the wave operator $(\partial^2/\partial t^2 - \Delta)$ and the oscillatory integral theorem of Carleson-Sjölin and Stein.

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