Calderon Reproducing Formula in Harmonic Analysis

Yongsheng Han

Auburn University, Auburn, Alabama, USA

A. Past (60's to 80's): The Calderon reproducing formula on R^n

(1) The continuous version of Calderon reproducing formula:

Suppose $\phi \in L^1(R^n)$ is real valued, radial and satisfies the following conditions:
(i) $Supp\phi \subset \{x \in R^n : |x| \le 1\}$;
(ii) $\phi \in C^\infty(R^n)$;
(iii) $\int_0^\infty [\widehat{\phi}(t\xi)]^2 \frac{dt}{t} = 1$ if $\xi \in R^n \backslash \{0\}$.
Then, if $f \in L^2(R^n)$,

$$f(x) = \int_0^\infty \phi_t * \phi_t * f(x) \frac{dt}{t}.$$

(a) Calderon, 1964: Intermediate spaces and interpolation, the complex method, Studia math. 24 (1964), 113-190.

(b) Peetre, 1976: New thoughts on Besov spaces, Duke Univ. Math. series, Durham, NC.

(c) Calderon, 1977: An atomic decomposition of distributions in parabolic H^p spaces, Adv. in Math. 25 (1977), 216-225.

(d) Chang and Fefferman, 1980: A continuous version of duality of H^1 and BMO on the bidisc, Ann. of Math. 112 (1980), 51-80.

(e) Uchiyama, 1982: A constructive proof of the Fefferman-Stein decomposition of $BMO(R^n)$, Acta. Math. 148 (1982), 215-241.

(f) David and Journe, 1984: A boundedness criterion for generalized Calderon-Zygmund operators, Ann. Math. 120 (1984), 371-397.

(g) Grossman and Morlet, 1984: Decompositions of hardy functions into square integrable wavelets of constant shape, SIAM J. Math. Anal. 15 (1984), 723-736.

(2) The discontinuous version of Calderon reproducing formula:

Suppose ϕ satisfies the following conditions:

(i) $Supp\widehat{\phi} \subseteq \{\xi \in R^n : \frac{1}{2} \leq |\xi| \leq 2\}$;
(ii) $|\widehat{\phi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.
Then there exists $\psi \in \mathcal{S}$ satisfying the same conditions as ϕ such that

$$f(x) = \sum_{\nu \in Z} \sum_{k \in Z^n} 2^{-\nu n} \psi_{2^{-\nu}}(x - k2^{-\nu}) \widetilde{\phi}_{2^{-\nu}} * f(k2^{-\nu})$$

$$= \sum_{Q} < f, \phi_Q > \psi_Q$$

where the series converges in $\mathcal{S}'/\mathcal{P}(R^n)$ and $L^2(R^n)$, $\widetilde{\phi}(x) = \phi(-x)$, and $\{Q\}$ is the collection of all dyadic cubes in R^n.

(a) Frazier and Jawerth, 1985: Decomposition of Besov spaces, Indiana Univ. math. J. 34 (1985), 777-799.

(b) Daubechies 1988: Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1988), 909-996.

(c) Frazier and Jawerth, 1990: A discrete transform and decomposition of distribution spaces, J. Funct. Anal. 93 (1990), 34-170.

(d) Meyer 1990: Ondelettes et operateurs, I, II, III, Hermann ed., Paris.

B. Present (90-): The Calderon Reproducing Formula on homogeneous and inhomogeneous spaces

Let X be a set. A quasi-metric defined on X is a function $\rho : X \times X :\longrightarrow [0, \infty)$ such that
(i) $\rho(x, y) = 0$ iff $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(iii) There exists a constant K such that for all x, y and $z \in X$,

$$\rho(x, y) \leq K[\rho(x, z) + \rho(z, y)].$$

For $r > 0$, Define $B(x, r) = \{y \in X : \rho(y, x) < r\}$.

$(X, \rho, d\mu)$ is said to be a space of homogeneous type if $d\mu$ is a nonnegative measure defined on X and satisfies the doubling condition: There exists a constant A such that

$$\mu(B(x, 2r)) \leq A\mu(B(x, r))$$

for all $x \in X$ and all $r > 0$.

$(R^n, d\mu)$ is said to be an inhomogeneous space if $d\mu$ is a nonnegative measure defined on R^n and satisfies the following growth condition: There exists a constant A and $0 < d \leq n$ such that for all $x \in R^n$ and all $r > 0$,

$$\mu(B(x,r)) \leq Ar^d$$

where $B(x,r) = \{y \in R^n : |x-y| < r\}$.

(1) The Calderon-type reproducing formula on spaces of homogeneous type:

Coifman's approximation to the identity: There exists an approximation to the identity $\{S_k\}$ defined on $X \times X$ such that
(i) $S_k(x,y) = 0$ if $\rho(x,y) \geq c2^{-k}$ and $|S_k(x,y)| \leq c2^k$;
(ii) $|S_k(x,y) - S_k(x',y)| \leq c\rho(x,x')^\epsilon 2^{(1+\epsilon)k}$;
(iii) $|S_k(x,y) - S_k(x,y')| \leq c\rho(y,y')^\epsilon 2^{(1+\epsilon)k}$;
(iv) $\int S_k(x,y)d\mu(y) = \int S_k(x,y)d\mu(y) = 1$.

The construction of $\{S_k\}$: Take a smooth function $h : R^+ \to R^+$ equal to 1 on $[0, \frac{1}{2}]$ and to 0 on $[2, \infty)$. Let

$$T_k(f)(x) = \int 2^k h(2^k \rho(x,y)) f(y) d\mu(y).$$

The doubling condition and properties of ρ imply that

$$\frac{1}{C} \leq T_k(1) \leq C$$

for some $0 < C < \infty$.

Let

$$M_k(f)(x) = [\frac{1}{T_k(1)(x)}]f(x)$$

and

$$W_k(f)(x) = [T_k(\frac{1}{T_k(1)})(x)]^{-1} f(x).$$

Finally,

$$S_k(f)(x) = M_k T_k W_k T_k M_k(f)(x).$$

It is easy to check that the kernel of S_k satisfy the conditions (i)-(iii). To see the condition (iv), since $S_k(x,y) = S_k(y,x)$, it is enough to see the first equality in (iv).

$$S_k(1)(x) = \int S_k(x,y)d\mu(y) = M_k T_k W_k T_k M_k(1)$$

$$= M_k T_k W_k T_k \frac{1}{T_k(1)(x)}$$

$$= M_k T_k [T_k(\frac{1}{T_k(1)})(x)]^{-1}[T_k(\frac{1}{T_k(1)})(x)]$$

$$= [\frac{1}{T_k(1)(x)}]T_k(1)(x) = 1.$$

Coifman's idea: By (i)-(iv), $S_k \to I$, the identity in L^2, as $k \to \infty$, and $S_k \to 0$ as $k \to -\infty$. let $D_k = S_{k+1} - S_k$. Then,

$$I = \sum_k D_k = \sum_k D_k \sum_j D_j$$

$$= \sum_{|k-j| \le N} D_k D_j + \sum_{|k-j|>N} D_k D_j = T_N + R_N.$$

David, Journe and Semmes proved that R_N is bounded on L^p for $1 < p < \infty$, and on BMO, and the operator norm of R_N is less than 1. Thus, $(T_N)^{-1}$, the inverse of T_N, is also bounded on $L^p, 1 < p < \infty$, and on BMO. The Calderon-type reproducing formula is given by

$$f = (T_N)^{-1} T_N(f) = (T_N)^{-1} \sum_k D_k^N D_k(f)$$

$$= \sum_k [(T_N)^{-1} D_k^N] D_k(f).$$

Using this formula, David, Journe and Semmes established the Littlewood-Paley theory on $L^p, 1 < p < \infty$, and proved the Tb theorem for space of homogeneous type.

(2) The Calderon reproducing formula on spaces of homogeneous type:

Definition: Fix two exponents $0 < \beta \le 1$ and $\gamma > 0$, A function f defined on X is said to be a test function of type (β, γ, x_0, d), if f satisfies the following conditions:
(i) $|f(x)| \le C \frac{d^\gamma}{(d+\rho(x,x_0))^{1+\gamma}}$,
(ii) $|f(x) - f(x')| \le C[\frac{\rho(x,x')}{d+\rho(x,x_0)}]^\beta \frac{d^\gamma}{(d+\rho(x,x_0))^{1+\gamma}}$
for $\rho(x,x') \le \frac{1}{2K}(d + \rho(x,x_0)$.
(iii) $\int f(x) d\mu(x) = 0$.

$$\|f\|_{\mathcal{M}(\beta,\gamma,x_0,d)} = inf\{C \ge 0 : (i), (ii) and (iii) hold\}.$$

It is easy to check that $\mathcal{M}(\beta, \gamma, x_0, d)$ is a Banach space. We denote by $(\mathcal{M}(\beta, \gamma, x_0, d))'$ the dual space of $\mathcal{M}(\beta, \gamma, x_0, d)$.

Definition: A continuous complex-valued function $K(x, y)$ defined on $\{(x, y) \in X \times X : x \ne y\}$ is called a C-Z kernel if there exist $\epsilon > 0$ and a constant $C < \infty$ such that for all $x, y \in X$ with $x \ne y$,

(i) $|K(x,y)| \le C\rho(x,y)^{-1}$,
(ii) $|K(x,y)-K(x,y')| + |K(y,x)-K(y',x)| \le C\rho(y,y')^{\epsilon}\rho(x,y)^{-(1+\epsilon)}$
for all x, y and $y' \in X$ with $\rho(y,y') \le \frac{1}{2K}\rho(x,y)$.

Theorem([Han]): If T is a continuous operator from $C_0^{\eta}(X)$ to $(C_0^{\eta}(X))'$ for $\eta > 0$ with a C-Z kernel, T is bounded on L^2 and $T(1) = T^*(1) = 0$. Moreover, $K(x,y)$ satisfies the following second difference smoothness condition:

$$|[K(x,y)-K(x,y')] - [K(x',y)-K(x',y')]|$$

$$\le C\rho(x,x')^{\epsilon}\rho(y,y')^{\epsilon}\rho(x,y)^{-(2+\epsilon)}$$

for all x, x', y and $y' \in X$ with $\rho(x,x') \le \frac{1}{2K^2}\rho(x,y)$ and $\rho(y,y') \le \frac{1}{2K^2}\rho(x,y)$.

Then T maps $\mathcal{M}(\beta,\gamma,x_0,d)$ to $\mathcal{M}(\beta,\gamma,x_0,d)$ with $0<\beta<\epsilon$ and $0<\gamma<\epsilon$.

We need the following definition of an approximation to the identity:

The approximation to the identity: A family of operators $\{S_k\}$ is said to be an approximation to the identity if $S_k(x,y)$, the kernel of S_k satisfy the following conditions:
(i) $|S_k(x,y)| \le C\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{1+\epsilon}}$;
(ii) $|S_k(x,y)-S_k(x',y)| \le C[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}]^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{1+\epsilon}}$
for $\rho(x,x') \le \frac{1}{2K}[2^{-k}+\rho(x,y)]$;
(iii) $|S_k(x,y)-S_k(x,y')| \le C[\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}]^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{1+\epsilon}}$
for $\rho(y,y') \le \frac{1}{2K}[2^{-k}+\rho(x,y)]$;
(iv) $|[S_k(x,y)-S_k(x',y)]-[S_k(x,y')-S_k(x',y')| \le C[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}]^{\epsilon}[\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}]^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{1+\epsilon}}$
for $\rho(x,x') \le \frac{1}{2K^2}[2^{-k}+\rho(x,y)]$ and $\rho(y,y') \le \frac{1}{2K^2}[2^{-k}+\rho(x,y)]$;
(v) $\int S_k(x,y)d\mu(y) = \int S_k(x,y)d\mu(x) = 1$.

The continuous Calderon reproducing formula on spaces of homogeneous type([Han and Sawyer]):

Suppose $\{S_k\}$ is an approximation to the identity. Let $D_k = S_{k+1} - S_k$. Then there exist $\{\widetilde{D}_k\}$ and $\{\overline{D}_k\}$ such that for all $f \in \mathcal{M}(\beta,\gamma,x_0,d)$,

$$f(x) = \sum_k \widetilde{D}_k D_k(f)(x) = \sum_k D_k \overline{D}_k(f)(x)$$

where the series converges in $\mathcal{M}(\beta',\gamma',x_0,d)$ with $0<\beta'<\beta$ and $0<\gamma'<\gamma$. For all $f \in (\mathcal{M}(\beta,\gamma,x_0,d))'$ the series also converges in $(\mathcal{M}(\beta',\gamma',x_0,d))'$ with $0<\beta<\beta'$ and $0<\gamma<\gamma'$. More over, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfy the following estimates: for any $0<\epsilon'<\epsilon$, there exists a constant C such that
(i) $|\widetilde{D}_k(x,y)| \le C\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{1+\epsilon'}}$,
(ii) $|\widetilde{D}_k(x,y)-\widetilde{D}_k(x',y)| \le C[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}]^{\epsilon'}\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{1+\epsilon'}}$,
for all x and x' with $\rho(x,x') \le \frac{1}{2K}[2^{-k}+\rho(x,y)]$,

(iii) $\int \widetilde{D}_k(x,y)d\mu(y) = \int \widetilde{D}_k(x,y)d\mu(x) = 0$.

$\overline{D}_k(x,y)$, the kernel of $\overline{D}_k$, satisfy the same conditions with interchanging x and y.

We now recall a construction of Christ, which provides an analogue of the grid of Euclidean dyadic cubes on a space of homogeneous type.

Theorem([Christ]): Given $\delta > 0$ sufficiently small, there exists a collection of open subsets $\{Q_\nu^k \subseteq X : k \in Z, \nu \in I_k\}$, where I_k denotes some(possibly finite) index set depending on k, and positive constants a, η, and C such that

(i) $\mu(X \backslash \bigcup_\nu Q_\nu^k) = 0$ for all k,

(ii) If $j \geq k$ then either $Q_{\nu'}^j \subseteq Q_\nu^k$ or $Q_{\nu'}^j \cap Q_\nu^k = \phi$,

(iii) For each (k,ν) and each $j < k$, there is a unique ν' such that $Q_\nu^k \subseteq Q_{\nu'}^j$,

(iv) Diameter $(Q_\nu^k) \leq C\delta^k$,

(v) Each Q_ν^k contains some ball $B(z_\nu^k, a\delta^k)$.

The discrete Calderon reproducing formula on spaces of homogeneous type([Han]):

Suppose $\{S_k\}$ is an approximation to the identity. Let $D_k = S_{k+1} - S_k$. Then there exist $\{\widetilde{D}_k\}$ and $\{\overline{D}_k\}$ such that for all $f \in \mathcal{M}(\beta,\gamma,x_0,d)$,

$$f(x) = \sum_k \sum_{\nu \in k+N} \mu(Q_\nu^{k+N}) \widetilde{D}_k(x, y_\nu^{k+N}) D_k(f)(y_\nu^{k+N})$$

$$= \sum_k \sum_{\nu \in k+N} \mu(Q_\nu^{k+N}) D_k(x, y_\nu^{k+N}) \overline{D}_k(f)(y_\nu^{k+N})$$

where N is a large integer, y_ν^{k+N} are any fixed point in Q_ν^{k+N}, and the series converges in $\mathcal{M}(\beta',\gamma',x_0,d)$ with $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$. For all $f \in (\mathcal{M}(\beta,\gamma,x_0,d))'$ the series also converges in $(\mathcal{M}(\beta',\gamma',x_0,d))'$ with $0 < \beta < \beta'$ and $0 < \gamma < \gamma'$. More over, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfy the following estimates: for any $0 < \epsilon' < \epsilon$, there exists a constant C such that

(i) $|\widetilde{D}_k(x,y)| \leq C\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{1+\epsilon'}}$,

(ii) $|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \leq C[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}]^{\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{1+\epsilon'}}$,

for all x and x' with $\rho(x,x') \leq \frac{1}{2K}[2^{-k} + \rho(x,y)]$,

(iii) $\int \widetilde{D}_k(x,y)d\mu(y) = \int \widetilde{D}_k(x,y)d\mu(x) = 0$.

$\overline{D}_k(x,y)$, the kernel of $\overline{D}_k$, satisfy the same conditions with interchanging x and y.

The idea of the proof:

$$f(x) = \sum_k \widetilde{D}_k D_k(f)(x) = \sum_k \sum_{\nu \in I_{k+N}} \int_{Q_\nu^{k+N}} \widetilde{D}_k(x,y) D_k(f)(y) d\mu(y)$$

$$= \sum_k \sum_{\nu \in I_{k+N}} \mu(Q_\nu^{k+N}) \widetilde{D}_k(x, y_\nu^{k+N}) D_k(f)(y_\nu^{k+N}) +$$

$$\{ \sum_k \sum_{\nu \in I_{k+N}} \int_{Q_\nu^{k+N}} [\widetilde{D}_k(x,y) - \widetilde{D}_k(x, y_\nu^{k+N})] D_k(f)(y) d\mu(y) +$$

$$\sum_k \sum_{\nu \in I_{k+N}} \int_{Q_\nu^{k+N}} \widetilde{D}_k(x, y_\nu^{k+N}) [D_k(f)(y) - D_k(f)(y_\nu^{k+N})] d\mu(y) \}$$

$$= T_N + R_N.$$

It can be shown that R_N satisfies theorem with the norm less than $C2^{-N\delta}$ for some $\delta > 0$. If N is chosen large enough, then $(T_N)^{-1}$ maps test function to test function. Thus,

$$f(x) = (T_N)^{-1} T_N(f)(x) =$$

$$\sum_k \sum_{\nu \in k+N} \mu(Q_\nu^{k+N}) ((T_N)^{-1} \widetilde{D}_k)(x, y_\nu^{k+N}) D_k(f)(y_\nu^{k+N}).$$

Applications:

(A) Function spaces on spaces of homogeneous type([Han and Sawyer], [Han]):

Besov spaces: For $|s| < \epsilon, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < p \le \infty, 0 < q \le \infty$,

$$\dot{B}_p^{s,q}(X) = \{ f \in (\mathcal{M}(\beta,\gamma))' : \|f\|_{\dot{B}_p^{s,q}} = \{ \sum_k (2^{ks} \|D_k(f)\|_p)^q \}^{\frac{1}{q}} < \infty \}.$$

Triebel-Lizorkin spaces: For $|s| < \epsilon, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < p < \infty, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < q \le \infty$,

$$\dot{F}_p^{s,q}(X) = \{ f \in (\mathcal{M}(\beta,\gamma))' : \|f\|_{\dot{F}_p^{s,q}} = \| \{ \sum_k (2^{ks} |D_k(f)|)^q \}^{\frac{1}{q}} \|_p < \infty \}.$$

To see these definitions are independent of the choice of D_k, we consider $p, q > 1$ case and we need to show

$$\{ \sum_k (2^{ks} \|D_k(f)\|_p)^q \}^{\frac{1}{q}} \le C \{ \sum_j (2^{js} \|E_j(f)\|_p)^q \}^{\frac{1}{q}}.$$

In fact,

$$\|D_k(f)\|_p = \|D_k(\sum_j \widetilde{E}_j E_j(f))\|_p \le \sum_j \|D_k \widetilde{E}_j\|_{pp} \|E_j(f))\|_p.$$

The key estimate: For any $0 < \epsilon' < \epsilon$, there exists a constant C such that

$$\|D_k \widetilde{E}_j\|_{pp} \le C 2^{-|k-j|\epsilon'}.$$

Thus,

$$\{\sum_k (2^{ks}\|D_k(f)\|_p)^q\}^{\frac{1}{q}} \le C\{\sum_k (\sum_j 2^{ks}2^{-|k-j|\epsilon'}\|E_j(f)\|_p)^q\}^{\frac{1}{q}}$$

$$\le C\{\sum_k (\sum_j 2^{(k-j)s}2^{-|k-j|\epsilon'}2^{js}\|E_j(f)\|_p)^q\}^{\frac{1}{q}} \le C\{\sum_j (2^{js}\|E_j(f)\|_p)^q\}^{\frac{1}{q}}.$$

To deal with the case where $min\{p,q\} \le 1$, we need the so-called Plancherel-Polya type inequality. Idea of this inequality comes from the following estimates:

$$\{\sum_k (2^{ks}\|D_k(f)\|_p)^q\}^{\frac{1}{q}} =$$

$$\{\sum_k 2^{ksq}(\sum_\nu \int_{Q_\nu^{k+N}} |D_k(f)(x)|^p d\mu(x))^{\frac{q}{p}}\}^{\frac{1}{q}}$$

$$\simeq \{\sum_k 2^{ksq}(\sum_\nu \int_{Q_\nu^{k+N}} |D_k(f)(y_{Q_\nu^{k+N}})|^p d\mu(x))^{\frac{q}{p}}\}^{\frac{1}{q}}$$

where y_ν^{k+N} is any fixed point in Q_ν^{k+N}

$$\simeq \{\sum_k 2^{ksq}(\sum_\nu \mu(Q_\nu^{k+N})|D_k(f)(y_{Q_\nu^{k+N}})|^p)^{\frac{q}{p}}\}^{\frac{1}{q}}$$

$$\simeq \{\sum_k 2^{ksq}(\sum_\nu \mu(Q_\nu^{k+N})|D_k(f)(y_{Q_\nu^{k+N}})|^p)^{\frac{q}{p}}\}^{\frac{1}{q}}$$

$$\simeq \{\sum_k 2^{k(s-\frac{1}{p})q}(\sum_\nu |D_k(f)(y_{Q_\nu^{k+N}})|^p)^{\frac{q}{p}}\}^{\frac{1}{q}}.$$

Similarly,

$$\|\{\sum_k (2^{ks}|D_k(f)|)^q\}^{\frac{1}{q}}\|_p =$$

$$\|\{\sum_k \sum_\nu \chi(Q_\nu^{k+N})(x)(2^{ks}|D_k(f)(x)|)^q\}^{\frac{1}{q}}\|_p$$

$$\simeq \|\{\sum_k \sum_\nu \chi(Q_\nu^{k+N})(x)(2^{ks}|D_k(f)(y_\nu^{k+N})|)^q\}^{\frac{1}{q}}\|_p$$

$$\simeq \|\{\sum_k \sum_\nu 2^{ksq}|D_k(f)(y_\nu^{k+N})|^q\chi(Q_\nu^{k+N})(x)\}^{\frac{1}{q}}\|_p.$$

The Plancherel-Ploya type inequality gives more precise estimates.

The Plancherel-Polya type inequality([Han]):

For $|s| < \epsilon, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < p \le \infty, 0 < q \le \infty$,

$$\{\sum_k 2^{k(s-\frac{1}{p})q}(\sum_\nu \sup_{z\in Q_\nu^{k+N}} |D_k(f)(z)|^p)^{\frac{q}{p}}\}^{\frac{1}{q}}$$

$$\simeq \{\sum_k 2^{k(s-\frac{1}{p})q}(\sum_\nu \inf_{z\in Q_\nu^{k+N}} |D_k(f)(z)|^p)^{\frac{q}{p}}\}^{\frac{1}{q}}.$$

For $|s| < \epsilon, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < p < \infty, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < q \leq \infty$,

$$\|\{\sum_k \sum_\nu 2^{ksq} \sup_{z\in Q_\nu^{k+N}} |D_k(f)(z)|^q \chi(Q_\nu^{k+N})(x)\}^{\frac{1}{q}}\|_p$$

$$\simeq \|\{\sum_k \sum_\nu 2^{ksq} \inf_{z\in Q_\nu^{k+N}} |D_k(f)(z)|^q \chi(Q_\nu^{k+N})(x)\}^{\frac{1}{q}}\|_p.$$

The proof of the Plancherel-Polya type inequality follows from the discrete Calderon reproducing formula. Using this inequality, one we can show definitions of the Besov and Triebel-Lizorkin spaces are independent of the choice of D_k. Also, using Calderon reproducing formula, we can obtain atomic decompositions, dual spaces, and other results about Besov and Triebel-Lizorkin spaces.

(B) The boundedness of C-Z operators on Besov and Triebel-Lizorkin spaces

Theorem: Suppose T is an operator from C_0^η to $(C_0^\eta)'$ with the kernel $K(x,y)$ satisfies

(i) $|K(x,y)| \leq C\rho(x,y)^{-1}$,

(ii) $|K(x,y) - K(x',y)| \leq C\rho(x,x')^\epsilon \rho(x,y)^{-(1+\epsilon)}$

for $\rho(x,x') \leq \frac{1}{2K}\rho(x,y)$,

and $T(1) = 0$, and T has the weak boundedness property:

$$< T(f), g >\leq Cr^{1+2\eta}\|f\|_\eta \|g\|_\eta$$

for all $f, g \in C_0^\eta$ with $suppf, suppg \subseteq B(x_0, r), \|f\|_\infty \leq 1, \|f\|_\eta \leq r^{-\eta}, \|g\|_\infty \leq 1, \|g\|_\eta \leq r^{-\eta}$.

Then T is bounded on $\dot{B}_p^{s,q}(X)$ for $0 < s < \epsilon, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < p \leq \infty, 0 < q \leq \infty$, and on $\dot{F}_p^{s,q}(X)$ for $0 < s < \epsilon, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < p < \infty, max\{\frac{1}{1+\epsilon+s}, \frac{1}{1+\epsilon}\} < q \leq \infty$.

The idea of the proof is the following: Consider the case where $min\{p,q\} > 1$.

$$\|Tf\|_{\dot{B}_p^{s,q}} = \{\sum_k (2^{ks}\|D_k(Tf)\|_p)^q\}^{\frac{1}{q}} =$$

$$\{\sum_k (2^{ks}\|D_k(T\sum_j D_j\overline{D}_j(f))\|_p)^q\}^{\frac{1}{q}} =$$

$$\{\sum_k (2^{ks}\|\sum_j (D_kTD_j)\overline{D}_j(f))\|_p)^q\}^{\frac{1}{q}}.$$

The key estimate: For any $0 < \epsilon' < \epsilon$ there exists a constant C such that

$$\|D_kTD_j\|_{pp} \leq C2^{-|k-j|\epsilon'}.$$

Repeating the above proof yields

$$\|Tf\|_{\dot{B}_p^{s,q}} \le C\{\sum_k(\sum_j 2^{(k-j)s}2^{-|k-j|\epsilon'}2^{js}\|\overline{D}_j(f)f)\|_p)^q\}^{\frac{1}{q}} \le$$

$$C\{\sum_j(2^{js}\|\overline{D}_j(f)f)\|_p)^q\}^{\frac{1}{q}} \le C\|f\|_{\dot{B}_p^{s,q}}.$$

(C) Hardy space and the Tb theorem

A Calderon-Zygmund singular integral operator T is a continuous linear operator from $C_0^\eta(R^n), \eta > 0$, into its dual associated to a kernel $K(x,y)$, a continuous function defined on $R^n \times R^n \backslash \{x = y\}$, satisfying the following conditions: there exist a constant C and $0 < \epsilon \le 1$, such that
(i) $|K(x,y)| \le C|x-y|^{-n}$,
(ii) $|K(x,y) - K(x',y)| \le C|x-x'|^\epsilon|x-y|^{-(n+\epsilon)}$
for $|x-x'| \le \frac{1}{2}|x-y|$,
(iii) $|K(x,y) - K(x,y')| \le C|y-y'|^\epsilon|x-y|^{-(n+\epsilon)}$
for $|y-y'| \le \frac{1}{2}|x-y|$,
(iv) $< Tf, g >= \int\int K(x,y)f(y)g(x)dydx$
for all $f, g \in C_0^\eta$ with $supp f \cap supp g = \phi$.

The Calderon-Zygmund theory says that if T is bounded on $L^2(R^n)$, then T is bounded on $L^p(R^n)$ for $1 < p < \infty$, from L^∞ to BMO, from $H^p(R^n)$, $\frac{n}{n+1} < p \le 1$, to $L^p(R^n)$. If $T^*(1) = 0$, then T is bounded on $H^p(R^n)$ for $\frac{n}{n+1} < p \le 1$. What happens if $T^*(b) = 0$, where b is a para-accretive function: b is said to be para-accretive if there exist constant $C, \gamma > 0$ such that, for all cubes $Q \subseteq R^n$, there is a $Q' \subseteq Q$ with $\gamma|Q| \le |Q'|$ satisfying

$$\frac{1}{|Q|}|\int_{Q'} b(x)dx| \ge C > 0.$$

The most important example is the Cauchy integral on a Lipschitz curve:

$$C(f)(x) = p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x - y + i(a(x) - a(y))}dy$$

where $a' \in L^\infty(R)$.

It was well known that C is a Calderon-Zygmund singular integral operator and is bounded on $L^2(R)$. But C is not bounded on $H^1(R)$ because $C(1) \ne 0$. However, $C(b) = 0$ if $b(y) = 1 + a'(y)$.

In fact, if T is a Calderon-Zygmund singular integral operator and is bounded on $L^2(R^n)$, and $T^*(b) = 0$, where b is a para-accretive function, then T is bounded from $H^p(R^n)$ to $H_b^p(R^n)$.

To establish $H_b^p(R^n)$, we need the Calderon reproducing formula associated to b.

Approximation to the identity of David, Journe and Semmes: Given a para-accretive function b, there exists an approximation to the identity $\{S_k\}$ such that $S_k(x,y)$, the kernel of S_k satisfy the following conditions:
(i) $|S_k(x,y)| \le C\frac{2^{-k\epsilon}}{(2^{-k}+|x-y|)^{1+\epsilon}}$;
(ii) $|S_k(x,y) - S_k(x',y)| \le C[\frac{|x-x'|}{2^{-k}+|x-y|}]^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+|x-y|)^{1+\epsilon}}$
for $|x-x'| \le \frac{1}{2}[2^{-k}+|x-y|]$;
(iii) $|S_k(x,y) - S_k(x,y')| \le C[\frac{|y-y'|}{2^{-k}+|x-y|}]^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+|x-y|)^{1+\epsilon}}$
for $|y-y'| \le \frac{1}{2}[2^{-k}+|x-y|]$;
(iv) $\int S_k(x,y)b(y)dy = \int S_k(x,y)b(x)dx = 1$.
But we proved that $S_k(x,y)$ still satisfy:
(v) $|[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')| \le C[\frac{|x-x'|}{2^{-k}+|x-y|}]^{\epsilon}[\frac{|y-y'|}{2^{-k}+|x-y|}]^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+|x-y|)^{1+\epsilon}}$
for $|x-x'| \le \frac{1}{2}[2^{-k}+|x-y|]$ and $|y-y'| \le \frac{1}{2}[2^{-k}+|x-y|]$.

Calderon reproducing formula associated to a para-accretive function b:

Suppose that $\{S_k\}$ is an approximation to the identity associated to a para-accretive function b. Set $D_k = S_{k+1} - S_k$. Then there exist $\{\widetilde{D}_k\}$ and $\{\widetilde{\widetilde{D}}_k\}$ such that for all $f \in \mathcal{M}_b(\beta,\gamma,x_0,d)$,

$$f(x) = \sum_k \widetilde{D}_k b D_k b(f)(x) = \sum_k D_k b \widetilde{\widetilde{D}}_k b(f)(x)$$

where the series converges in $\mathcal{M}_b(\beta',\gamma',x_0,d)$ with $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$. More over, $\widetilde{D}_k(x,y)$, the kernel of $\widetilde{D}_k$, satisfy the following estimates: for any $0 < \epsilon' < \epsilon$, there exists a constant C such that
(i) $|\widetilde{D}_k(x,y)| \le C\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{1+\epsilon'}}$,
(ii) $|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}]^{\epsilon'}\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(x,y))^{1+\epsilon'}}$,
for all x and x' with $\rho(x,x') \le \frac{1}{2K}[2^{-k}+\rho(x,y)]$,
(iii) $\int \widetilde{D}_k(x,y)b(y)dy = \int \widetilde{D}_k(x,y)b(x)dx = 0$.
$\widetilde{\widetilde{D}}_k(x,y)$, the kernel of $\widetilde{\widetilde{D}}_k$, satisfy the same conditions with interchanging x and y.

The Hardy space $H_b^p(R^n)$ is defined by

$$H_b^p = \{f \in (b\mathcal{M}_b(\beta,\gamma))' : \|g_b(f)\|_p < \infty\}$$

where

$$g_b(f)(x) = \{\sum_k |D_k bf(x)|^2\}^{\frac{1}{2}}.$$

Theorem([Han, Lee and Lin]): Suppose T is a Calderon-Zygmund singular integral operator, is bounded on L^2 and $T^*(b) = 0$, where b is a para-accretive function. Then T is bonded from H^p to H^p_b for $\frac{n}{n+\epsilon} < p \le 1$, where ϵ depends on b.

(D) Wavelet Frame Theory

A wavelet frame on L^2 is a family $\{\psi_{j,k}(x) = 2^{jn/2}\psi(2^j x - k)\}$ such that there exist two constants C_1 and C_2 for all $f \in L^2$,

$$C_1\|f\|_2 \le \{\sum_{j,k} | < \psi_{j,k} f > |^2\}^{\frac{1}{2}} \le C_2\|f\|_2.$$

A necessary condition for wavelet frame is the following inequality: If $\{\psi_{j,k}\}$ is a frame on L^2, then there exist constants A and B such that

$$0 < A \le \int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} \le B < \infty \qquad (*)$$

for almost all $\xi \in R^n$.

In some sense, the following theorem gives a necessary and sufficient condition for wavelet frame:

Theorem ([GHHLWW]): If $\psi \in \mathcal{M}(\beta, \gamma)$ for $\beta, \gamma > 0$ and satisfies the condition $(*)$, then there exist a_0 and b_0 such that $\{\psi_{j,k}(x) = a^{jn/2}\psi(a^n x - bk)\}$ is a frame on $L^2(R^n)$ for all $1 < a \le a_0$ and $0 < b \le b_0$.

Idea of the proof: Define

$$T(f)(x) = \int_0^\infty \psi_t * \psi_t * f(x)\frac{dt}{t}.$$

Then we can show $(T)^{-1}$, the inverse of T, maps test function to test function. So we obtain the following Calderon reproducing formula:

$$f(x) = (T)^{-1}\int_0^\infty \psi_t * \psi_t * f(x)\frac{dt}{t} = \int_0^\infty (T)^{-1}\psi_t * \psi_t * f(x)\frac{dt}{t}$$

$$= \int_0^\infty \theta_t * \psi_t * f(x)\frac{dt}{t}$$

where $\theta = (T)^{-1}\psi$.

(3) The Calderon-type-reproducing formula on inhomogeneous spaces:

Approximation to the identity of Tolsa: There exists a family of operators $\{S_k\}$ with kernels $S_k(x, y)$ satisfying the following conditions:

(i) $S_k(x,y) = S_k(y,x)$,
(ii) $\int S_k(x,y)d\mu(y) = 1$ for all x,
(iii) If $Q_{x,k}$ is a transit cube, then $supp(S_k(x,.)) \subseteq Q_{x,k-1}$,
(iv) If $Q_{x,k}$ and $Q_{y,k}$ are transit cubes,

$$0 \leq S_k(x,y) \leq \frac{C}{((Q_{x,k}) + (Q_{y,k}) + |x-y|)^n},$$

(v) If $Q_{x,k}, Q_{x'}$ and $Q_{y,k}$ are transit cubes, and $x, x' \in Q_{x0,k}$ for some x_0, then

$$|S_k(x,y) - S_k(x',y)| \leq C\frac{|x-x'|}{(Q_{x0,k})}\frac{1}{((Q_{x,k}) + (Q_{y,k}) + |x-y|)^n}.$$

Tolsa proved the Calderon-type reproducing formula: for all $f \in L^2(R^n)$,

$$f = \sum_k [(T_N)^{-1} D_k^N] D_k(f).$$

Using this formula, Tolsa gives another proof of the $T1$ theorem for inhomogeneous spaces.

Deng, Han and Yang proved this formula holds on a new test function spaces and its dual.

New test function spaces: for $1, p, q \leq \infty$,

$$\dot{\mathcal{B}}_p^{s,q}(R^n) =$$

$$\{f \in L^2(R^n) : \|f\|_{\dot{\mathcal{B}}_p^{s,q}} = \{\sum_k (2^{ks}\|D_k(f)\|_p)^q\}^{\frac{1}{q}} < \infty\}.$$

For $1 < p, q < \infty$,

$$\dot{\mathcal{F}}_p^{s,q}(R^n) =$$

$$\{f \in L^2(R^n) : \|f\|_{\dot{\mathcal{F}}_p^{s,q}} = \|\{\sum_k (2^{ks}|D_k(f)|)^q\}^{\frac{1}{q}}\|_p < \infty\}.$$

First we need to show that these spaces are independent of the choice of D_k. The idea of the proof is the following: Suppose D_k and E_j come from Tolsa's approximation to the identity. Then by Tolsa's Calderon-type reproducing formula, for f$\in L^2(R^n)$,

$$D_k(f)(x) = D_k \sum_j [(T_N)^{-1} E_j^N] E_j(f)(x)$$

where

$$I = T_N + R_N, (T_N)^{-1} = \sum_{i=0}^{\infty} (R_N)^i,$$

and

$$R_N = \sum_{|-j|>N} E_j E.$$

Put all these together,

$$D_k(f)(x) =$$
$$\sum_{i=0}\sum_j \sum_{|_1-j_1|>N}\sum_{|_2-j_2|>N}\cdots$$
$$\sum_{|_i-j_i|>N} D_kE_{j_1}E_{_1}E_{j_2}E_{_2}...E_{j_i}E_{_i}E_j^NE_j(f)(x).$$

Thus,

$$\|D_k(f)\|_p \leq$$
$$\sum_{i=0}\sum_j \sum_{|_1-j_1|>N}\sum_{|_2-j_2|>N}\cdots$$
$$\sum_{|_i-j_i|>N} \|D_kE_{j_1}E_{_1}E_{j_2}E_{_2}...E_{j_i}E_{_i}E_j^N\|_{pp}\|E_j(f)\|_p.$$

Finally, use Cotlar-Stein's technique to estimate

$$\|D_kE_{j_1}E_{_1}E_{j_2}E_{_2}...E_{j_i}E_{_i}E_j^N\|_{pp}.$$

Denote the dual spaces of $\dot{\mathcal{B}}_p^{s,q}(R^n)$ and $\dot{\mathcal{F}}_p^{s,q}(R^n)$ by $(\dot{\mathcal{B}}_p^{s,q}(R^n))'$ and $(\dot{\mathcal{F}}_p^{s,q}(R^n))'$, respectively.

We need the second difference smoothness condition for Tolsa's approximation to the identity. In fact, we proved

(vi) If $Q_{x,k}, Q_{x'}, Q_{y,k}$ and $Q_{y',k}$ are transit cubes, and $x, x' \in Q_{x_0,k}$ for some x_0, and $y, y' \in Q_{y_0,k}$ for some y_0, then

$$|[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')]|$$
$$\leq C\frac{|x-x'|}{(Q_{x_0,k})}\frac{|y-y'|}{(Q_{y_0,k})}\frac{1}{((Q_{x,k}) + (Q_{y,k}) + |x-y|)^n}.$$

The Calderon-type reproducing formula for $\dot{\mathcal{B}}_p^{s,q}(R^n), \dot{\mathcal{F}}_p^{s,q}(R^n), (\dot{\mathcal{B}}_p^{s,q}(R^n))'$ and $(\dot{\mathcal{F}}_p^{s,q}(R^n))'$

$$f = \sum_k (T_N)^{-1}D_k^ND_k(f) = \sum_k D_kD_k^N(T_N)^{-1}(f)$$

where the series converge in $\dot{\mathcal{B}}_p^{s,q}(R^n), \dot{\mathcal{F}}_p^{s,q}(R^n), (\dot{\mathcal{B}}_p^{s,q}(R^n))'$ and $(\dot{\mathcal{F}}_p^{s,q}(R^n))'$.

Conclusion

As Coifman said: " The objective of Zygmund, Calderon and their school was not the establishment of new theorems by any means possible. It was often to take known results that seemed like magic-e.g. because of the way they used complex variables methods-and tear them apart, finding the underlying structures and their inner interaction that made it absolutely clear what was going on. The test of understanding was measured by the ability to prove an estimate."

(C) Future

Three questions:

(1) The Hardy space associated to non doubling measures.

(2) The Hardy space associated to flag singular integrals.

Example: On $R^2, K(x,y) = \frac{1}{x(x+iy)}$ and consider

$$Tf = p.v.K * f.$$

(3) The Hardy space associated to non-standard dilations.

Example: On $R^3, \delta_1^2\delta_2^2 K(\delta_1 x, \delta_2 y, \delta_1\delta_2 z) = K(x,y,z)$ and consider

$$Tf = p.v.K * f.$$