ON GENERALIZED RIESZ POTENTIALS AND SPACES OF SOME SMOOTH FUNCTIONS

EIICHI NAKAI AND HIRONORI SUMITOMO

ABSTRACT. Let \((X, \delta, \mu)\) be a normal space of homogeneous type of order \(\gamma\). Gatto and Vágı [6] showed that, if \(f\) and \(I_\alpha f\) are in \(L^p(X) (0 < \alpha < \min(\gamma, 1/p))\), then \(I_\alpha f\) is in \(C^p,\alpha(X)\), where \(C^p,\alpha\) is the space of smooth functions of Calderón-Scott [1].

In this paper, we introduce a generalized Riesz potential \(I_\phi\) and extend the result above. With this aim, we extend the Hardy-Sobolev inequality to the Orlicz space.

1. Introduction

Let \(X = (X, d, \mu)\) be a space of homogeneous type, i.e. \(X\) is a topological space endowed with a quasi-distance \(d\) and a positive measure \(\mu\) such that

\[
d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y,
\]

\[
d(x, y) = d(y, x),
\]

\[
d(x, y) \leq K_1 (d(x, z) + d(z, y)),
\]

the balls \(B(x, r) = \{y \in X : d(x, y) < r\}\), \(r > 0\), form a basis of neighborhoods of the point \(x\), \(\mu\) is defined on a \(\sigma\)-algebra of subsets of \(X\) which contains the balls, and

\[
0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty,
\]

where \(K_i \geq 1 \quad (i = 1, 2)\) are constants independent of \(x, y, z \in X\) and \(r > 0\).

We assume that \(X = (X, d, \mu)\) is of order \(\gamma \quad (0 < \gamma \leq 1)\) and \(Q\)-homogeneous \((Q > 0)\), i.e.

\[
|d(x, z) - d(y, z)| \leq K_3 d(x, y)^\gamma (d(x, z) + d(y, z))^{1-\gamma},
\]

\[
K_4^{-1} r^Q \leq \mu(B(x, r)) \leq K_4 r^Q,
\]

where \(K_i \geq 1 \quad (i = 3, 4)\) are constants independent of \(x, y, z \in X\) and \(r > 0\).

The \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) is of order \(1\) and \(n\)-homogeneous.

For an increasing function \(\phi : (0, \infty) \rightarrow (0, \infty)\), let

\[
I_\phi f(x) = \int_X f(y) \frac{\phi(d(x, y))}{d(x, y)^Q} d\mu(y).
\]

If \(\phi(r) = r^\alpha, 0 < \alpha < Q\), then \(I_\phi\) is the Riesz potential of order \(\alpha\).
For \( f \in L^p(X) \), \( 1 < p < \infty \), we consider the sharp functions
\[
\frac{f(x)}{\alpha} = \sup_{x \in B(a,r)} \frac{1}{\alpha} \int_{B(a,r)} |f(y) - f_{\alpha}(a,r)| \, d\mu(y)
\]
where \( f_{\alpha}(a,r) = \mu(B(a,r))^{-1} \int_{B(a,r)} f(y) \, d\mu(y) \) and the supremum is taken over all balls \( B(a,r) \) containing \( x \).

The space \( C^{p,\phi}(X) \) is the set of all functions \( f \in L^p(X) \) with \( f \in L^p(X) \) equipped with the norm \( \|f\|_{C^{p,\phi}} = \|f^{\alpha}\|_p + \|f\|_p \), where \( \|\cdot\|_p \) denotes the \( L^p \)-norm.

Our main results are as follows:

**Theorem 1.1.** Let \( 1 < p < \infty \), \( 1/p + 1/p' = 1 \). Assume that \( \phi \) is increasing, \( \phi(r)/r^{(Q/p-\varepsilon)} \) is decreasing for some \( \varepsilon > 0 \), and \( \int_0^1 \phi(t)/t \, dt + \int_1^\infty (\phi(t)/t^{1+\gamma}) \, dt < \infty \). Let
\[
\psi(r) = \int_0^r \frac{\phi(t)}{t} \, dt + r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt, \quad 0 < r < \infty.
\]
If \( f \) and \( F = I_\phi f \) are in \( L^p(X) \), then \( F \) is in \( C^{p,\phi}(X) \) and \( \|F\|_{C^{p,\phi}} \leq C(\|f\|_p + \|f\|_p) \) with a constant \( C \) independent of \( F \) and \( f \).

**Remark 1.1.** If \( \phi \) is increasing and \( \phi(r)/r^Q \) is decreasing, then \( \phi \) is continuous and
\[
\phi(r) \leq \psi(2r) \leq 2^Q \psi(r),
\]
\[
\phi(r) \leq C_Q \left( \int_0^r \frac{\phi(t)}{t} \, dt + r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt \right).
\]

**Corollary 1.2.** Let \( 1 < p < \infty \), \( 1/p + 1/p' = 1 \). Assume that \( \phi \) is increasing, \( \phi(r)/r^{(Q/p-\varepsilon)} \) is decreasing for some \( \varepsilon > 0 \), and there is a constant \( C_0 > 0 \) such that
\[
\int_0^r \frac{\phi(t)}{t} \, dt + r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt \leq C_0 \psi(r), \quad 0 < r < \infty.
\]
If \( f \) and \( F = I_\phi f \) are in \( L^p(X) \), then \( F \) is in \( C^{p,\phi}(X) \) and \( \|F\|_{C^{p,\phi}} \leq C(\|f\|_p + \|f\|_p) \) with a constant \( C \) independent of \( F \) and \( f \).

**Remark 1.2.** If \( \phi(r) = r^\alpha \) (\( 0 < \alpha < \gamma \)), then \( \phi \) satisfies (1.5). Therefore the result of [6, Theorem 2.1] is contained in this corollary.

To prove the results above, we extend the Hardy-Sobolev inequality to the Orlicz space \( L^\Phi \). The definitions of the N-function \( \Phi \) and the Orlicz space \( L^\Phi \) are in next section.

**Theorem 1.3.** Let \( 1 < s < \infty \). Assume that \( \phi \) is increasing, \( \phi(r)/r^{(Q/s-\varepsilon)} \) is decreasing for some \( \varepsilon > 0 \), and \( \int_0^1 \phi(t)/t \, dt < \infty \). Then there is an N-function \( \Psi \) such that
\[
C^{-1} \phi^{-1} \left( \frac{1}{r^Q} \right) \leq \frac{1}{r^Q/s} \int_0^r \frac{\phi(t)}{t} \, dt \leq C \phi^{-1} \left( \frac{1}{r^Q} \right), \quad 0 < r < \infty,
\]
and \( I_\phi \) is bounded from \( L^s(X) \) to \( L^\Phi(X) \).

Section 3 is for preliminalies. In Section 4 we give proofs of the theorems. In Section 5 we give examples.

The letter \( C \) will denote a constant, not necessarily the same indifferent occurrences.
2. Orlicz spaces

In this section, we recall the definition of Orlicz spaces. A function \( \Phi : [0, \infty) \to [0, \infty) \) is called an N-function if it can be represented as

\[
\Phi(r) = \int_0^r a(t) \, dt,
\]

where \( a : [0, \infty) \to [0, \infty) \) is a left continuous nondecreasing function such that \( a(0) = 0 \) and \( a(t) \to \infty \) as \( t \to \infty \). Let

\[
b(r) = \inf \{ s : a(s) > r \}.
\]

Then

\[
\Psi(r) = \int_0^r b(t) \, dt
\]

is also an N-function, and \((\Phi, \Psi)\) is called a complementary pair.

Let \((X, \mu)\) be a measure space. For an N-function \( \Phi \), let

\[
L^\Phi(X) = \left\{ f : \int_X \Phi(\varepsilon |f(x)|) \, d\mu(x) < \infty \text{ for some } \varepsilon > 0 \right\},
\]

\[
\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_X \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.
\]

Let \((\Phi, \Psi)\) be a complementary pair of N-functions. We note that

\[
\int_X |f(x)g(x)| \, d\mu(x) \leq 2\|f\|_\Phi \|g\|_\Psi,
\]

and that

\[
r \leq \Phi^{-1}(r)\Psi^{-1}(r), \quad r \geq 0,
\]

where \( \Phi^{-1} \) and \( \Psi^{-1} \) are inverse functions of \( \Phi \) and \( \Psi \), respectively. Let \((X, d, \mu)\) be a space of homogeneous type, and \( \chi_{B(a, r)} \) be the characteristic function of a ball \( B(a, r) \). Then

\[
\|\chi_{B(a, r)}\|_\Psi = \inf \left\{ \lambda > 0 : \int_X \Psi \left( \frac{\chi_{B(a, r)}(x)}{\lambda} \right) \, d\mu(x) \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \Psi \left( \frac{1}{\lambda} \right) \mu(B(a, r)) \leq 1 \right\}
\]

\[
= \frac{1}{\Psi^{-1}(1/\mu(B(a, r)))} \leq \mu(B(a, r))\Phi^{-1} \left( \frac{1}{\mu(B(a, r))} \right).
\]

3. Preliminaries

In this section, we show lemmas to prove theorems.

**Lemma 3.1.** Let \( \alpha > 0, \beta > 0, \delta > 0, \phi : (0, \infty) \to (0, \infty) \) be increasing and \( \phi(r)/r^\alpha \) be decreasing. Then, for \( 0 < r < \infty \),

\[
\left( \frac{1}{(\alpha + \beta)\delta} \right)^{1/\delta} \phi(r)^{1/\delta} \leq \int_r^\infty \left( \frac{\phi(t)^{\delta}}{t^{\alpha + \beta}} \right)^{1/\delta} \, dt \leq \left( \frac{1}{\beta\delta} \right)^{1/\delta} \phi(r)^{1/\delta}.
\]
Proof. By the increasingness of $\phi$ we have
\[
\int_{r}^{\infty} \left( \frac{\phi(t)}{t^{\alpha+\beta}} \right)^{\delta} t^{-1} dt = \int_{r}^{\infty} \phi(t)^{\delta} t^{-1-(\alpha+\beta)\delta} dt \\
\geq \phi(r)^{\delta} \int_{r}^{\infty} t^{-1-(\alpha+\beta)\delta} dt = \frac{1}{(\alpha+\beta)\delta} \left( \frac{\phi(r)}{r^{\alpha+\beta}} \right)^{\delta}.
\]
By the decreasingness of $\phi(r)/r^\alpha$ we have
\[
\int_{r}^{\infty} \left( \frac{\phi(t)}{t^{\alpha+\beta}} \right)^{\delta} t^{-1} dt = \int_{r}^{\infty} \left( \frac{\phi(t)}{t^\alpha} \right)^{\delta} t^{-1-\beta\delta} dt \\
\leq \left( \frac{\phi(r)}{r^\alpha} \right)^{\delta} \int_{r}^{\infty} t^{-1-\beta\delta} dt = \frac{1}{\beta\delta} \left( \frac{\phi(r)}{r^{\alpha+\beta}} \right)^{\delta}.
\]

Lemma 3.2. Let $\alpha > 0$, $\beta > 0$, $\alpha + \beta < Q$, $h : (0, \infty) \to (0, \infty)$ be increasing and differentiable, and $h(r)/r^\alpha$ be decreasing. Then there is an $N$-function $\Phi$ such that
\[
C^{-1} \Phi^{-1} \left( \frac{1}{r^Q} \right) \leq \frac{h(r)}{r^{\alpha+\beta}} \leq C \Phi^{-1} \left( \frac{1}{r^Q} \right), \quad 0 < r < \infty,
\]
where $C > 0$ is independent of $r$.

Proof. Let
\[
\Phi^{-1} \left( \frac{1}{r^Q} \right) = \int_{r}^{\infty} \frac{h(t)}{t^{\alpha+\beta}} t^{-1} dt.
\]
Applying Lemma 3.1 with $\delta = 1$, we have (3.1). Next we show $\Phi'(u) > 0$. Let
\[
u = \Phi^{-1} \left( \frac{1}{r^Q} \right), \quad v = \frac{1}{r^Q}.
\]
Then $v = \Phi(u)$ and
\[
\frac{dv}{du} = \frac{du}{dr} \frac{dv}{dr} = \left( -\frac{Q}{r^{Q+1}} \right) \left( -\frac{h(r)}{r^{\alpha+\beta+1}} \right) = \frac{Q}{r^{Q-\alpha-\beta} h(r)}
\]
is strictly decreasing with respect to $r$. Hence
\[
\frac{d^2 v}{du^2} = \left( \frac{dv}{du} \right) \frac{du}{dr} > 0.
\]

Remark 3.1. If $\phi$ is increasing and $\phi(r)/r^\alpha$ is decreasing, then $h(r) = \int_{r}^{\infty} (\phi(t)/t) dt$ is increasing and differentiable, and $h(r)/r^\alpha$ is decreasing. Actually,
\[
\frac{d}{dr} \left( \frac{h(r)}{r^\alpha} \right) = \frac{r h'(r) - \alpha h(r)}{r^{\alpha+1}} = \frac{1}{r^{\alpha+1}} \left( \phi(r) - \alpha \int_{0}^{r} \frac{\phi(t)}{t^\alpha} t^{-1} dt \right)
\]
\[
\leq \frac{1}{r^{\alpha+1}} \left( \phi(r) - \alpha \frac{\phi(r)}{r^\alpha} \int_{0}^{r} t^{-1} dt \right) = 0.
\]

Lemma 3.3. Let $\phi$ be increasing and $\phi(r)/r^Q$ be decreasing. If $2K_1 d(x, x') \leq d(x, y)$, then
\[
\left| \phi(d(x, y)) \left( \frac{d(x, y)}{d(x', y)} \right)^{Q} \phi(d(x', y)) \right| \leq C d(x, x') \phi(d(x, y)) \left( \frac{d(x', y)}{d(x, y)} \right)^{Q\gamma},
\]
where $C > 0$ is independent of $x, x', y \in X$. 

Proof. By mean value theorem, for \( u < r_0 < v \), we have
\[
0 \leq \frac{\phi(u)}{u^Q} - \frac{\phi(v)}{v^Q} \leq \phi(u)\left(\frac{1}{u^Q} - \frac{1}{v^Q}\right) = \phi(u)(v - u) \frac{d}{dr}\left(-\frac{1}{r^{Q}}\right)_{r=r_0} = Q\phi(u)(v - u) \frac{1}{r_0^{Q+1}} \leq Q(v - u) \frac{\phi(u)}{u^{Q+1}}.
\]

Let \( u = \min(d(x, y), d(x', y)) \) and \( v = \max(d(x, y), d(x', y)) \). Then
\[
v - u \leq K_3 d(x, x')^\gamma (d(x, y) + d(x', y))^{1-\gamma} \leq K_3 \left( K_1 + \frac{3}{2} \right)^{1-\gamma} d(x, x')^\gamma d(x, y)^{1-\gamma},
\]
and
\[
\frac{d(x, y)}{2K_1} \leq u \leq d(x, y).
\]

Hence
\[
(v - u)\frac{\phi(u)}{u^{Q+1}} \leq Cd(x, x')^\gamma \frac{\phi(d(x, y))}{d(x, y)^{Q+\gamma}}.
\]

Therefore we have (3.2). \( \square \)

The following is used in the proof of Theorem 1.1. For all balls \( B \) and for all integrable functions \( f \) on \( B \),
\[
\frac{1}{\mu(B)} \int_B |f(y) - f_B| \, d\mu(y) \leq 2 \inf_c \frac{1}{\mu(B)} \int_B |f(y) - c| \, d\mu(y).
\]

4. PROOFS OF THEOREMS

Proof of Theorem 1.3. By Lemma 3.2 and Remark 3.1 we have an N-function \( \Phi \) with the property (1.6). For \( r > 0 \), let
\[
J_1 = \int_{d(x, y) < r} f(y) \frac{\phi(d(x, y))}{d(x, y)^Q} \, d\mu(y) \quad \text{and} \quad J_2 = \int_{d(x, y) \geq r} f(y) \frac{\phi(d(x, y))}{d(x, y)^Q} \, d\mu(y).
\]

Since \( \phi(r)/r^Q \) is decreasing,
\[
|J_1| \leq Mf(x) \int_{d(x, y) < r} \frac{\phi(d(x, y))}{d(x, y)^Q} \, d\mu(y),
\]
where \( M \) is the Hardy-Littlewood maximal function (see Stein[9, p.57]). By (1.2) and (1.4) we have
\[
\frac{\phi(d(x, y))}{d(x, y)^Q} \, d\mu(y) \leq \frac{\phi(r_j)}{r_j^Q} \mu(B(x, 2r_j)) \leq C \phi(r_j) \leq C' \int_{r_j}^{2r_j} \frac{\phi(t)}{t} \, dt, \quad r_j = 2^{-j}r, \; j = 1, 2, \ldots.
\]

From (4.1) and (4.2) it follows that
\[
|J_1| \leq CMf(x) \int_0^r \frac{\phi(t)}{t} \, dt.
\]
Next we estimate $|J_2|$. Let $1/s + 1/s' = 1$. Let $\chi_{B(x,r)}$ be the characteristic function of $B(x, r)$. By Hölder inequality we have

$$|J_2| \leq \|f\|_s \left\| \frac{\phi(d(x, \cdot))}{d(x, \cdot)^Q} \chi_{B(x,r)}(\cdot) \right\|_{s'} = \|f\|_s \left( \int_{d(x,y) \geq r} \left( \frac{\phi(d(x,y))}{d(x,y)^Q} \right)^{s'} \mu(y) \right)^{1/s'}.$$

By (1.2) and (1.4) we have

$$\int_{r_j \leq d(x,y) < 2r_j} \left( \frac{\phi(d(x,y))}{d(x,y)^Q} \right)^{s'}\mu(y) \leq \left( \frac{\phi(r_j)}{r_j^Q} \right)^{s'} \mu(B(x, 2r_j)) \leq C \left( \frac{\phi(r_j)}{r_j^Q} \right)^{s'} \leq C' \int_{r_j}^{2r_j} \left( \frac{\phi(t)}{t^Q} \right)^{s'} t^{-1} dt, \quad r_j = 2^j r, \; j = 0, 1, 2, \ldots.$$

By Lemma 3.1 we have

$$\left( \int_{r}^{\infty} \left( \frac{\phi(t)}{t^Q} \right)^{s'} t^{-1} dt \right)^{1/s'} \leq C \frac{\phi(r)}{r^{Q/s}}.$$

From (4.4), (4.5) and (4.6) it follows that

$$|J_2| \leq C \|f\|_s \frac{\phi(r)}{r^{Q/s}}.$$

By (4.3) and (4.7) we have

$$|I_\phi f(x)| \leq C \left( Mf(x) + \|f\|_s \frac{1}{r^{Q/s}} \right) \int_0^r \frac{\phi(t)}{t} dt.$$
Proof of Theorem 1.1. Fix $x \in X$; we will estimate $F\psi^s_\phi(x)$. Let $B = B(a, r)$ be a ball containing $x$ and $\tilde{B} = B(a, 2r)$. Let $\chi$ be the characteristic function of $\tilde{B}$. Set $F = F_1 + F_2$ with $F_1 = I_\phi(f\chi)$ and $F_2 = I_\phi(f(1 - \chi))$.

To estimate $(F_1)^s_\psi^\psi(x)$, let $1 < s < p$. By Theorem 1.3 we have an N-function $\Phi$ with the property (1.6) and

\[ (4.8) \quad \|I_\phi f\|_\Phi \leq C\|f\|_s. \]

Let $\Psi$ be the complement of $\Phi$. From (2.1), (2.3), (1.6), (1.4) and (4.8), it follows that

\[
\frac{1}{r^{Q\psi(r)}} \int_B |I_\phi(f\chi)(z)| \, d\mu(z) \leq \frac{2}{r^{Q\psi(r)}} \|\chi_B\|_\psi \|I_\phi(f\chi)\|_\psi
\]

\[
\leq \frac{2}{r^{Q\psi(r)}} \mu(B) \Phi^{-1} \left( \frac{\mu(B)}{1} \right) \|I_\phi(f\chi)\|_\psi \leq C \frac{1}{r^{Q\psi(r)}} \|f\chi\|_s
\]

\[
= C \left( \frac{1}{r^Q} \int_B |f(z)|^s \, d\mu(z) \right)^{1/s} \leq C' M_s(f)(x),
\]

where $M_s(f) = [M(|f|^s)]^{1/s}$. By (3.3) we have

\[ (4.9) \quad (F_1)^s_\psi^\psi(x) \leq C M_s(f)(x). \]

Second we estimate $(F_2)^s_\psi^\psi(x)$. Observe that

\[
I_\phi(f(1 - \chi))(z) - I_\phi(f(1 - \chi))(a)
\]

\[
= \int_{(\tilde{B})^c} f(y) \left( \frac{\phi(d(z, y))}{d(z, y)^Q} - \frac{\phi(d(a, y))}{d(a, y)^Q} \right) \, d\mu(y),
\]

then by Lemma 3.3 we have

\[ (4.10) \quad \int_B |I_\phi(f(1 - \chi))(z) - I_\phi(f(1 - \chi))(a)| \, d\mu(z)
\]

\[
\leq C \int_{B} |z - a|^\gamma \left( \int_{(\tilde{B})^c} \frac{\phi(d(a, y))|f(y)|}{d(a, y)^{Q+\gamma}} \, d\mu(y) \right) \, d\mu(z).
\]

To estimate the inner integral we write

\[
\int_{(\tilde{B})^c} \frac{\phi(d(a, y))|f(y)|}{d(a, y)^{Q+\gamma}} \, d\mu(y) \leq \sum_{k=1}^\infty \int_{2^{k+r} \leq d(a, y) < 2^{k+1+r}} \frac{\phi(2^kr)|f(y)|}{(2^kr)^{Q+\gamma}} \, d\mu(y)
\]

\[
\leq \sum_{k=1}^\infty \frac{(2^{k+1+r})^Q}{(2^kr)^{Q+\gamma}} \frac{\phi(2^kr)}{(2^kr)^{Q+\gamma}} \int_{B(a, 2^{k+1+r})} |f(y)| \, d\mu(y)
\]

\[
\leq C \left( \sum_{k=1}^\infty \frac{\phi(2^kr)}{(2^kr)^\gamma} \right) Mf(x) \leq C' \left( \sum_{k=1}^\infty \int_{2^{k-1+r} \leq t < 2^{k+r}} \frac{\phi(t)}{t^{1+\gamma}} \, dt \right) Mf(x)
\]

\[
= C' \left( \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt \right) Mf(x) \leq C' \frac{\psi(r)}{r^\gamma} Mf(x).
\]

Using the estimate (4.10) and (3.3) we get

\[ (4.11) \quad (F_2)^s_\psi^\psi(x) \leq C Mf(x) \leq CM_s(f)(x). \]
By (4.9), (4.11) and the fact that the sharp function operator is subadditive, we have

\[ F^\sharp_\psi(x) \leq CM_*(f)(x). \]

Finally, using the strong type \( p/s \) of \( M \) we have

\[ \| F^\sharp_\psi \|_p \leq C\| f \|_p. \]

This concludes the proof of Theorem 1.1.

5. Examples

For functions \( \theta, \kappa : (0, \infty) \to (0, \infty) \), we denote \( \theta(r) \sim \kappa(r), u < r < v, \) if there is a constant \( C > 0 \) such that

\[ C^{-1}\theta(r) \leq \kappa(r) \leq C\theta(r), \quad u < r < v. \]

Let \( 0 \leq \alpha_i < \infty \) and \( -\infty < \beta_i < \infty \) \((i = 1, 2)\). For constants \( r_1 \) and \( r_2 \) \((0 < r_1 < 1/e, e < r_2)\), let

\[ \phi(r) = \begin{cases} k_1 r^{\alpha_1}(1/\log(1/r))^{\beta_1}, & 0 < r < r_1, \\ 1, & r_1 \leq r \leq r_2, \\ k_2 r^{\alpha_2}(\log r)^{\beta_2}, & r_2 < r < \infty, \end{cases} \]

where \( k_1 = (r_1^{\alpha_1}(1/\log(1/r_1)))^{\beta_1} \) and \( k_2 = (r_2^{\alpha_2}(\log r_2))^{\beta_2} \).

If \( \alpha_1, \alpha_2 > 0 \), then

\[ \int_0^r \frac{\phi(t)}{t} \, dt \sim \phi(r). \]

If \( \alpha_1, \alpha_2 < \gamma \), then

\[ r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt \sim \phi(r). \]

If \( \alpha_1 = 0, \beta_1 > 1 \), i.e. \( \phi(r) = k_1(1/\log(1/r))^{\beta_1}, 0 < r < r_1, \) then

\[ r^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt \sim \phi(r) \leq C \int_0^r \frac{\phi(t)}{t} \, dt = C'(1/\log(1/r))^{\beta_1-1}, \quad 0 < r < r_1, \]

i.e. \( \psi(r) \sim (1/\log(1/r))^{\beta_1-1}, 0 < r < r_1 \).

If \( \alpha_2 = \gamma, \beta_2 < -1 \), i.e. \( \phi(r) = r^\gamma(\log r)^{\beta_2}, r > r_2, \) then

\[ \int_0^r \frac{\phi(t)}{t} \, dt \sim \phi(r) \leq Cr^\gamma \int_r^\infty \frac{\phi(t)}{t^{1+\gamma}} \, dt = C' r^\gamma(\log r)^{\beta_2+1}, \quad r > r_2, \]

i.e. \( \psi(r) \sim r^\gamma(\log r)^{\beta_2+1}, r > r_2 \).

Let \( \alpha_1 = 0, 0 < \alpha_2 < \gamma, \beta_1 > 1 \). Choose \( r_1 \) and \( r_2 \) so that \( \phi \) is increasing and that \( (1/r)^n/p\phi(r) \) and \( (1/r)^n/p \int_0^r (\phi(t)/t) \, dt \) are decreasing. For \( x \in \mathbb{R}^n, 1 < \delta < p \), let

\[ f(x) = \begin{cases} (1/|x|)^n/p(1/\log(1/|x|))^{\delta/p}, & |x| < r_1, \\ 0, & |x| \geq r_1. \end{cases} \]
Then $f \in L^p(\mathbb{R}^n)$. Let $\Phi$ and $\Phi_1$ be N-functions such that

$$\Phi^{-1}\left(\frac{1}{r^n}\right) \sim \frac{1}{r^{n/p}} \int_0^r \frac{\phi(t)}{t} dt \quad \text{and}$$

$$\Phi_1^{-1}\left(\frac{1}{r^n}\right) \sim \frac{1}{r^{n/p}} \phi(r).$$

Then $I_\phi f \in L^\Phi(\mathbb{R}^n) \setminus L^{\Phi_1}(\mathbb{R}^n)$. Actually Theorem 1.3 yeilds that $I_\phi f \in L^\Phi(\mathbb{R}^n)$. If $|x| < r_1/2$ and $|y| < |x|/2$, then $|x|/2 \leq |x - y| \leq 3|x|/2$ and $f(x) \sim f(x - y)$. Hence,

$$I_\phi f(x) \geq \int_{|y| \leq |x|/2} f(x - y) \frac{\phi(|y|)}{|y|^n} dy \geq C f'(x) \int_0^{|x|/2} \frac{\phi(|y|)}{|y|^n} dy \geq C' f(x)(1/\log(2/|x|))^{\beta_1 - 1}$$

$$\geq C''(1/|x|)^{n/p}(1/\log(1/|x|))^{\beta_1} \sim \Phi_1^{-1}\left(\frac{1}{|x|^n}\right), \quad |x| < r_1/2.$$

Since $\Phi_1(r) \leq \Phi_1(2r) \leq C \Phi_1(r)$, for all $\lambda > 0$, there is a constant $\lambda' > 0$ such that

$$\Phi_1\left(\frac{I_\phi f(x)}{\lambda}\right) \geq \frac{1}{\lambda'} \frac{1}{|x|^n}, \quad |x| < \frac{r_1}{2}.$$

Therefore $I_\phi f \notin L^{\Phi_1}(\mathbb{R}^n)$.

REFERENCES


Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582, Japan
E-mail address: enakai@cc.osaka-kyoiku.ac.jp
Telephone and facsimile: 0729-78-3424