A COMPLETION OF B. MOSSÉ'S UNILATERAL RECOGNIZABILITY THEOREM

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Abstract. The goal of this paper is to provide complete statement and proof for B. Mossé's unilateral recognizability theorem.

1. Introduction

Every map σ from a finite alphabet A, which consists of at least two letters, to the set A^+ of nonempty words over the alphabet A is called a *substitution* on A. Suppose that a substitution σ on A is *primitive*, i.e. there exists $k \in \mathbb{N} := \{1, 2, 3, \dots\}$ such that any pair $(a, b) \in A \times A$, a occurs in $\sigma^k(b)$. Suppose that the primitive substitution σ has a fixed point $u \in A^{\mathbb{Z}_+}$, where $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. For each $p \in \mathbb{N}$, set $E_p = \{0\} \cup \{|\sigma^p(u_{[0,n)})| : n \in \mathbb{N}\}$, whose elements are called the *natural p-cutting points*; see also [5, § 3], [1, § 3.4] and [6, § 7.2.1]. It is clear that $E_q \subsetneq E_p$ whenever $q > p$.

Definition 1.1 ([2, p. 530])**.** The substitution σ is said to be *(unilaterally) recognizable* if there exists $L \in \mathbb{N}$ such that $u_{[i,i+L)} = u_{[j,j+L)}, i \in E_1 \Rightarrow j \in E_1$.

The recognizability does not depend on the choice of the fixed point u. If $v \in A^{\mathbb{Z}_+}$ is another fixed point of σ, then the primitivity of σ guarantees that the language of u coincides with that of v.

The so-called Morse substitution: $a \mapsto ab, b \mapsto ba$ has a fixed point

$$u = abbabaabbaababbabaababbaabbabaab\dots$$

and then $E_1 = \{0, 2, 4, 6, \dots\}$. As is shown in [7, p. 109], the Morse substitution is recognizable with $L = 4$.

The recognizability is an important notion for primitive substitutions from viewpoints of associated subshifts. If the substitution σ is recognizable, then the unilateral subshift X_σ arising from σ has a Kakutani-Rohlin partition built on a *clopen* subset $\sigma(X_\sigma)$ of X_σ. It is trivial that Kakutani-Rohlin partitions have played crucial roles in investigation of dynamical systems. Proposition VI. 6 of [7] states that given a point $x \in X_\sigma$, the first return time of the point $\sigma(x)$ to the clopen subset $\sigma(X_\sigma)$ equals the length $|\sigma(x_0)|$ of the word $\sigma(x_0)$. This leads to a fact that the first return map on $\sigma(X_\sigma)$ is a topological factor of X_σ, which shows a self-similarity of X_σ if the substitution σ is injective on the alphabet A; see [7, Corollary VI. 8]. It is also a significant consequence of the recognizability that $\sigma(X_\sigma)$ is open; see [7, Proposition VI. 3] and [2, Lemme 2]. The recognizability is a premise of the celebrated theorem of [2], which characterizes eigenvalues and eigenfunctions of the subshift X_σ.

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In view of the above-mentioned facts among others, it is important to characterize a class of primitive substitutions with the recognizability. B. Mossé [5] gave a characterization for the *non*-recognizability, whose statement is written in French:

Theorem 1.2 ([5, Théorème 3.1]). *Soit σ une substitution primitive admettant un point fixe non périodique u. Pour que σ ne soit pas reconnaissable, il faut et il suffit que pour tout entier L, il existe un mot B de longueur L et deux éléments a et b de A tels que:*

— *le mot $\sigma(b)$ est un suffixe strict de $\sigma(a)$,*
— *les mots $\sigma(a)B$ et $\sigma(b)B$ apparaissent dans u avec le même 1-découpage de B.*

The following sentences would be an English translation of this statement.

> Let σ be a primitive substitution admitting an aperiodic fixed point u. So that σ is not recognizable, it is necessary and sufficient that for all $L \in \mathbb{N}$ there exists a word B of length L and two elements a and b of A such that
> — the word $\sigma(b)$ is a strict suffix of $\sigma(a)$,
> — the words $\sigma(a)B$ and $\sigma(b)B$ appear in u with the same 1-cutting of B.

See Definition 3.1 for the definition of "the same 1-cutting".

The statement of Theorem 1.2 is incomplete to characterize the non-recognizability. Try to show the sufficiency. The words $\sigma(a)B$ and $\sigma(b)B$ must occur at some positions $i, j \in E_1$ in u, respectively, so that $u_{i'} = a$, $u_{j'} = b$, $i = |\sigma(u_{[0,i')})|$ and $j = |\sigma(u_{[0,j')})|$ for some $i', j' \in \mathbb{Z}_+$.

The characterization should be formulated as follows.

Theorem 1.3. *Let σ be a primitive substitution on a finite alphabet A admitting an aperiodic fixed point u. Then the following are equivalent.*

(1) σ is not recognizable;
(2) for each $L \in \mathbb{N}$, there exist $i, j \in \mathbb{Z}_+$ such that
— *$\sigma(u_j)$ is a strict suffix of $\sigma(u_i)$;*
— *$\sigma(u_{i+k}) = \sigma(u_{j+k})$ for each integer k with $1 \leq k \leq L$.*

B. Mossé's proof [5] for this theorem would be difficult to completely follow, in particular, Part (4) in p. 332, to which Step 3 in the present proof of Theorem 1.3 corresponds. Moreover, no proofs can be found in recent textbooks [6, 3, 8], though a proof for the *bilateral* recognizability is written in [3, pp. 163-164]. This motivated the author to write the present article.

Excepting Step 3 in the proof of Theorem 1.3, everything which the reader will see in the present article is due to B. Mossé or other pioneers. Step 3 is an improvement by the present paper.

2. Preliminaries

We shall make terminology excepting that in the preceding section. The empty word is denoted by Λ. Put $A^* = A^+ \cup \{\Lambda\}$. We say that a word $w \in A^*$ occurs in a word $v \in A^*$ if there exist $p, s \in A^*$ such that $v = pws$. We then write $w \prec v$. More specifically, w is said to occurs at the position $|p| + 1$ in v. The position is called an occurrence of w in v. Let $|v|_w$ denote the number of occurrences of w in v. The words p and s are called a prefix and suffix of v, respectively. We then write $p \prec_{\mathrm{p}} v$ and $s \prec_{\mathrm{s}} v$, respectively. If $|p| < |v|$ (resp. $|s| < |v|$), then p (resp. s) is called a strict prefix (resp. suffix) of v, and then we write $p \prec_{\mathrm{sp}} v$ (resp. $s \prec_{\mathrm{ss}} v$). A power of a word $w \in A^*$ is a word of the form $\underbrace{ww \dots w}_{n \text{ times}}$ with some $n \in \mathbb{Z}_+$. The power is denoted by w^n. In particular, $w^0 = \Lambda$.

Let σ be a primitive substitution on A. The domain of σ is naturally extended to each of A^+ and $A^{\mathbb{Z}_+}$ by means of concatenation or juxtaposition of words. A recursive formula defines the powers σ^k for $k \in \mathbb{Z}_+$ on each of A^+ and $A^{\mathbb{Z}_+}$.

A nonnegative, square matrix M is said to be primitive if there exists $n \in \mathbb{N}$ such that M^n is positive, i.e. the entries of M^n are positive. The incidence matrix M_σ of the substitution σ is defined to be an $A \times A$ matrix whose (a, b)-entry equals $|\sigma(a)|_b$. Clearly, a substitution is primitive if and only if the incidence matrix associated with the substitution is primitive. Notice that every (a, b)-entry of ${M_\sigma}^n$ equals $|\sigma^n(a)|_b$.

Lemma 2.1. *There exist $\lambda > 0$ and $C > 0$ such that for all $a \in A$ and $n \in \mathbb{N}$,*

$$C^{-1}\lambda^n \le |\sigma^n(a)| \le C\lambda^n.$$

Proof. Let λ denote the Perron eigenvalue of M_σ, i.e. such an eigenvalue $\lambda > 0$ that the absolute value of any other eigenvalue is less than λ. In virtue of [4, Theorem 4.5.12], there exist sets $\{c_{a,b} > 0 : a, b \in A\}$ and $\{\rho_{a,b}(n) \in \mathbb{R} : a, b \in A, n \in \mathbb{N}\}$ such that for all $a, b \in A$ and $n \in \mathbb{N}$,

$$|\sigma^n(a)|_b = \{c_{a,b} + \rho_{a,b}(n)\}\lambda^n \text{ and } \lim_{n\to\infty} \rho_{a,b}(n) = 0.$$

Fix a letter $a \in A$. Since $\lambda^{-n}|\sigma^n(a)| > 0$ for all $n \in \mathbb{N}$ and $\lambda^{-n}|\sigma^n(a)| \to \sum_{b\in A} c_{a,b}$ $(n \to \infty)$, there exists $C_a > 0$ such that $C_a^{-1} \le \lambda^{-n}|\sigma^n(a)| \le C_a$ for all $n \in \mathbb{N}$. Taking $C = \max_{a\in A} C_a$, we obtain the desired conclusion. □

Suppose that $u := u_0u_1u_2 \dots \in A^{\mathbb{Z}_+}$ is a fixed point of σ, where $u_i \in A$ for each $i \in \mathbb{Z}_+$. Let $\mathcal{L}(u)$ denote the language of the sequence u, i.e.

$$\mathcal{L}(u) = \{u_iu_{i+1} \dots u_j | i, j \in \mathbb{Z}_+, i \le j\} \cup \{\Lambda\}.$$

Set $\mathcal{L}_k(u) = \{w \in \mathcal{L}(u) : |w| = k\}$. We say that a word $w \in A^*$ occurs at a position $i \in \mathbb{Z}_+$ in u if $u_{[i,i+|w|)} := u_iu_{i+1} \dots u_{i+|w|-1} = w$. The integer i is called an *occurrence* of the word w. In virtue of [1, Proposition 25], the fixed point u is *linearly recurrent* with a constant $K \in \mathbb{N}$, i.e. any word occurring in u occurs infinitely often in u and there exists $K \in \mathbb{N}$ such that the difference between two successive occurrences of any word $w \in \mathcal{L}(u)$ is less than $K|w|$. Theorem 24 (ii) of [1] guarantees that if an *aperiodic* sequence $u' \in A^{\mathbb{Z}_+}$ is linearly recurrent with a constant K, then the sequence u' is $(K+1)$-power free, i.e. $w \in A^+, w^N \in \mathcal{L}(u') \Rightarrow N \le K$. The following lemma was obtained earlier by [5, Théorème 2.4].

Lemma 2.2. *There exists $K \in \mathbb{N}$ such that the fixed point u is $(K+1)$-power free.*

3. A proof of Theorem 1.3

Definition 3.1. (1) A finite sequence $\{\alpha, \sigma^p(u_{i'}), \sigma^p(u_{i'+1}), \dots, \sigma^p(u_{i'+k-1}), \beta\}$ is called a *natural p-cutting* of $u_{[i,i+\ell)}$ if $\alpha \prec_{\mathrm{s}} \sigma^p(u_{i'-1})$, $\beta \prec_{\mathrm{p}} \sigma^p(u_{i'+k})$,

$$u_{[i,i+\ell)} = \alpha\sigma^p(u_{i'})\sigma^p(u_{i'+1}) \dots \sigma^p(u_{i'+k-1})\beta$$

and $i + |\alpha| = |\sigma^p(u_{[0,i')})|$.

(2) Suppose that a word w occurs at positions i and j in u. The word w is said to have the *same p-cutting* at the positions i and j if

$$(E_p \cap [i, i + |w| - 1]) + (j - i) = E_p \cap [j, j + |w| - 1].$$

Compare these definitions with the original ones in [5, § 3]. We do not exclude the possibility that $\alpha = \sigma^p(u_{i'-1})$, $\alpha = \Lambda$, $\beta = \sigma^p(u_{i'+k})$ or $\beta = \Lambda$. Not every $u_{[i,i+\ell)}$ has a natural p-cutting, because we require $k \geq 1$ in Definition 3.1 (1). It is not necessary that a natural p-cutting is uniquely determined for given i and ℓ in Definition 3.1 (1).

We now proceed to our proof of Theorem 1.3.

Proof of Theorem 1.3. It is enough to prove the implication (1) ⇒ (2), because the converse implication is obvious. Fix an integer $k > C^2\{C^2(K+1)+2\}$, where C (resp. K) is as in Lemma 2.1 (resp. Lemma 2.2). Assume that σ is not recognizable.

Step 1. It follows from Lemma 3.2 below that for each $p \in \mathbb{N}$, there exist integers $i_p \in E_1$, $j_p \notin E_1$, $i'_p, j'_p \geq 0$, $h_p, \ell_p \geq 1$ and words $\alpha_p, \gamma'_p \in A^*$, $\gamma_p \in A^+$ such that

— $u_{[i_p,i_p+\ell_p)} = u_{[j_p,j_p+\ell_p)}$;
— $u_{[i_p,i_p+\ell_p)}$ has a natural p-cutting:

$$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \ldots, \sigma^p(u_{i'_p+k-1})\};$$

— $u_{[j_p+|\alpha_p|,j_p+\ell_p)}$ has a natural p-cutting:

$$\{\gamma_p, \sigma^p(u_{j'_p}), \sigma^p(u_{j'_p+1}), \ldots, \sigma^p(u_{j'_p+h_p-1}), \gamma'_p\}.$$

Define $m_p = \min\{m \in \mathbb{N} : \alpha_p\gamma_p \prec_{\mathrm{s}} \sigma^p(u_{[j'_p-m,j'_p)})\}$. It follows from facts:

$$|\sigma^p(u_{[j'_p-m_p+1,j'_p)})| < |\alpha_p\gamma_p|, \quad |\gamma_p\sigma^p(u_{[j'_p,j'_p+h_p)})\gamma'_p| = |\sigma^p(u_{[i'_p,i'_p+k)}|;$$
$$\alpha_p \prec_{\mathrm{s}} \sigma^p(u_{i'_p-1}),\ \gamma_p \prec_{\mathrm{s}} \sigma^p(u_{j'_p-1}),\ \gamma'_p \prec_{\mathrm{p}} \sigma^p(u_{j'_p+h_p})$$

that $m_p < 2C^2 + 1$ and $kC^{-2} - 2 \leq h_p \leq kC^2$ for all $p \in \mathbb{N}$. Hence,

$$\{(m_p, h_p, u_{[i'_p-1,i'_p+k)}, u_{[j'_p-m,j'_p+h_p]}) : p \in \mathbb{N}\}$$

is a finite set, and so there exists an infinite set $I \subset \mathbb{N}$ such that the elements of

$$\{(m_p, h_p, u_{[i'_p-1,i'_p+k)}, u_{[j'_p-m,j'_p+h_p]}) :\ p \in I\}$$

are constant. It allows us to put $m = m_p$ and $h = h_p$ for any $p \in I$.

Step 2. Let $p, q \in I$ $(p < q)$ be arbitrary. We have two natural q-cuttings:

$$\{\gamma_q, \sigma^q(u_{j'_q}), \sigma^q(u_{j'_q+1}), \ldots, \sigma^q(u_{j'_q+h-1}), \gamma'_q\} \tag{3.1}$$

of a word occurring at the position $j_q + |\alpha_q|$ and

$$\{\sigma^{q-p}(\gamma_p), \sigma^q(u_{j'_q}), \sigma^q(u_{j'_q+1}), \ldots, \sigma^q(u_{j'_q+h-1}), \sigma^{q-p}(\gamma'_p)\} \tag{3.2}$$

of a word occurring at the position $j_q + |\alpha_q\gamma_q| - |\sigma^{q-p}(\gamma_p)|$. Assume that the natural q-cuttings are not the same. Then one of the inequalities $|\gamma_q| \neq |\sigma^{q-p}(\gamma_p)|$ and $|\gamma'_q| \neq |\sigma^{q-p}(\gamma'_p)|$ follows.

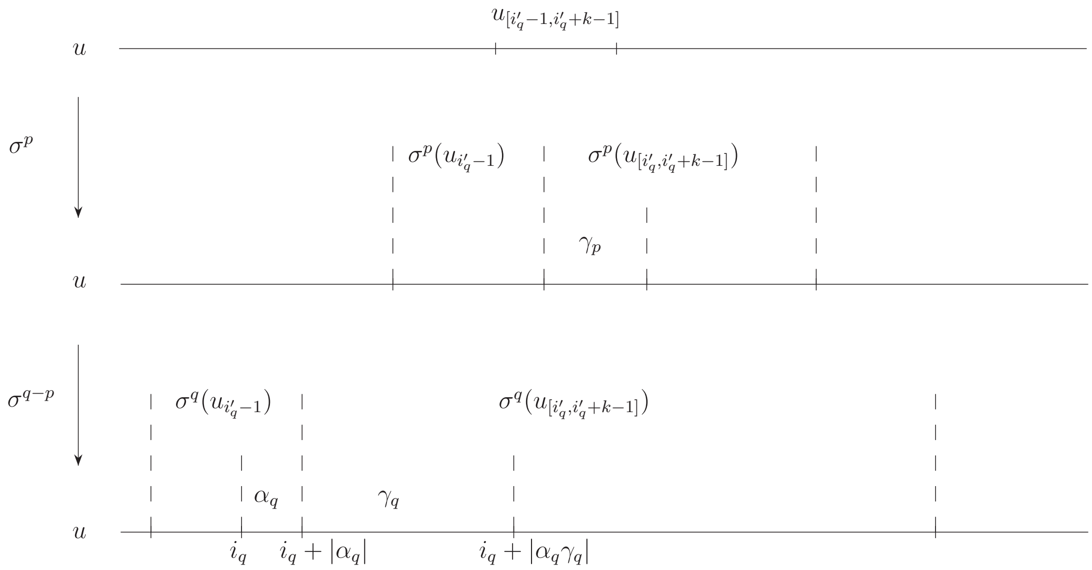

Figure 1

Consider the case $|\gamma_q| > |\sigma^{q-p}(\gamma_p)|$. This together with the fact that

$$\begin{aligned}\gamma_q\sigma^q(u_{[j'_q,j'_q+h)})\gamma'_q &= \sigma^q(u_{[i'_q,i'_q+k)})\\ &= \sigma^{q-p}(\sigma^p(u_{[i'_p,i'_p+k)}))\\ &= \sigma^{q-p}(\gamma_p\sigma^p(u_{[j'_p,j'_p+h)})\gamma'_p)\\ &= \sigma^{q-p}(\gamma_p)\sigma^q(u_{[j'_p,j'_p+h)})\sigma^{q-p}(\gamma'_p)\end{aligned}$$

implies that a power v^N of a nonempty word $v \prec_{\mathrm{ss}} \gamma_q$ occurs in $\sigma^q(u_{[j'_q,j'_q+h)})$ as a prefix. By using the fact that $v \prec_{\mathrm{s}} \sigma^q(u_{j'_q-1})$, we can see that

$$\max N \geq \frac{h\min_{a\in A}|\sigma^q(a)|}{\max_{a\in A}|\sigma^q(a)|} - 1 \geq (kC^{-2}-2)C^{-2} - 1 > K,$$

which contradicts Lemma 2.2. Since the same contradiction emerges in the other cases, we conclude that $\gamma_q = \sigma^{q-p}(\gamma_p)$ for any $p, q \in I$ with $p < q$.

Step 3. Choose $p, q \in I$ with $p < q$ so that $|\sigma^{q-1-p}(\gamma_p)| \geq L$.

Observe how $u_{[i'_q-1,i'_q+k)}$ goes to $\sigma^q(u_{[i'_q-1,i'_q+k)})$ via $\sigma^p(u_{[i'_q-1,i'_q+k)})$; see Figure 1. Since $\gamma_p \prec_{\mathrm{p}} \sigma^p(u_{[i'_q,i'_q+k)})$, $\gamma_q \prec_{\mathrm{p}} \sigma^q(u_{[i'_q,i'_q+k)})$ and $\sigma^{q-p}(\gamma_p) = \gamma_q$, we can see that $u_{[i_q+|\alpha_q|,i_q+|\alpha_q\gamma_q|)} = \gamma_q$ has a natural 1-cutting:

$$\{\sigma(u_{i''}), \sigma(u_{i''+1}), \ldots, \sigma(u_{i''+|\sigma^{q-1-p}(\gamma_p)|-1})\},$$

where $i'' = |\sigma^{q-1}(u_{[0,i'_q)})|$. Remark that $u_{[i'',i''+|\sigma^{q-1-p}(\gamma_p)|)} = \sigma^{q-1-p}(\gamma_p)$.

Then, observe how $u_{[j'_q-m,j'_q+h]}$ goes to $\sigma^q(u_{[j'_q-m,j'_q+h]})$ via $\sigma^p(u_{[j'_q-m,j'_q+h]})$; see Figure 2. Recalling that the natural q-cuttings (3.1) and (3.2) are the same, we can see that $u_{[j_q+|\alpha_q|,j_q+|\alpha_q\gamma_q|)} = \gamma_q$ has a natural 1-cutting:

$$\{\sigma(u_{j''}), \sigma(u_{j''+1}), \ldots, \sigma(u_{j''+|\sigma^{q-p-1}(\gamma_p)|-1})\},$$

where $j'' = |\sigma^{q-1-p}(u_{[0,|\sigma^p(u_{[0,j'_q)})|-|\gamma_p|)})|$. Remark that $u_{[j'',j''+|\sigma^{q-1-p}(\gamma_p)|)} = \sigma^{q-1-p}(\gamma_p)$.

We are finally in a situation that

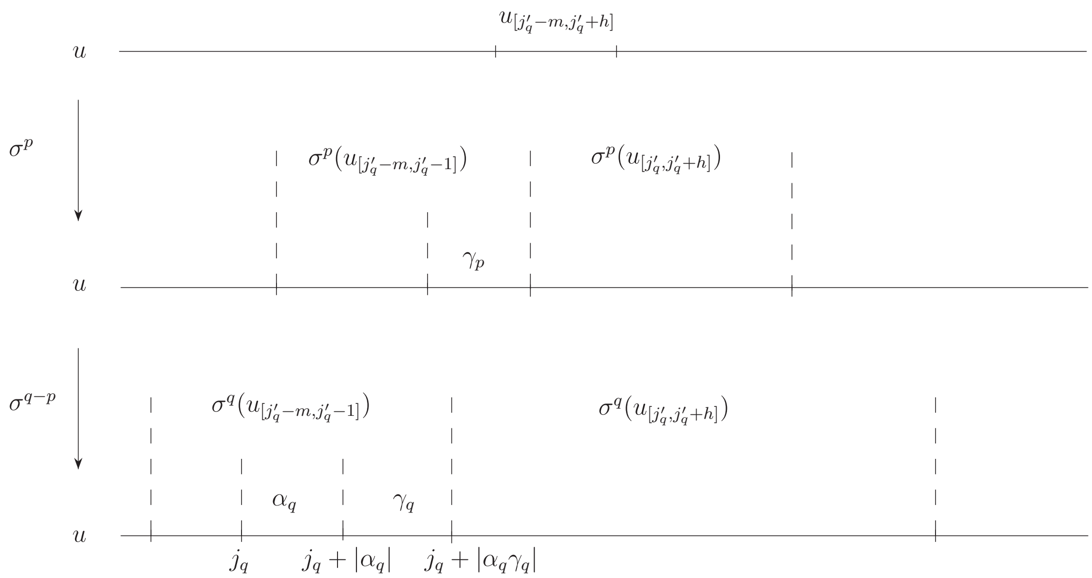

FIGURE 2

— $\alpha_q\gamma_q$ occurs at the positions $i_q \in E_1$ and $j_q \notin E_1$ in u;
— γ_q has the same 1-cutting at the positions $i_q + |\alpha_q|$ and $j_q + |\alpha_q|$;
— all of the positions $i_q + |\alpha_q|, i_q + |\alpha_q\gamma_q|, j_q + |\alpha_q|$ and $j_q + |\alpha_q\gamma_q|$ are natural 1-cutting points;
— the same 1-cutting of γ_q consists of at least L words.

We reach the desired positions $i, j \in \mathbb{Z}_+$ by means of the following procedure.

(P. 1) Set $\ell = i_q + |\alpha_q|$ and $m = j_q + |\alpha_q|$.
(P. 2) Let $\ell' < \ell$ and $m' < m$ be natural 1-cutting points which are nearest to ℓ and m respectively.
(P. 3) If $\ell - \ell' = m - m'$, then set $\ell = \ell'$ and $m = m'$. Go back to (P. 2).
(P. 4) In this step, we have that $\ell - \ell' \neq m - m'$. The desired positions i and j are determined by the facts that
  (a) $\ell - \ell' < m - m' \Rightarrow |\sigma(u_{[0,j)})| = \ell'$ and $|\sigma(u_{[0,i)})| = m'$;
  (b) $m - m' < \ell - \ell' \Rightarrow |\sigma(u_{[0,j)})| = m'$ and $|\sigma(u_{[0,i)})| = \ell'$.

The loop (P. 2) to (P. 3) continues up to $\left\lceil \dfrac{|\alpha_q\gamma_q|}{\min_{a\in A}|\sigma(a)|} \right\rceil$ times. This completes the proof. $\square$

Lemma 3.2. *Let C be a constant as in Lemma 2.1. Let $k \geq 3C^2$ be an integer. If the substitution σ is not recognizable, then for each $p \in \mathbb{N}$ there exist integers $i_p \in E_1, j_p \notin E_1, i'_p, j'_p \geq 0, h_p, \ell_p \geq 1$ and words $\alpha_p, \gamma'_p \in A^*, \gamma_p \in A^+$ such that*

— $u_{[i_p,i_p+\ell_p)} = u_{[j_p,j_p+\ell_p)}$*;*
— $u_{[i_p,i_p+\ell_p)}$ *has a natural p-cutting:*

$$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \dots, \sigma^p(u_{i'_p+k-1})\};$$

— $u_{[j_p+|\alpha_p|,j_p+\ell_p)}$ *has a natural p-cutting:*

$$\{\gamma_p, \sigma^p(u_{j'_p}), \sigma^p(u_{j'_p+1}), \dots, \sigma^p(u_{j'_p+h_p-1}), \gamma'_p\}.$$

Proof. Fix an integer $m_p > (k+2)\max_{a\in A}|\sigma^p(a)|$. Since σ is not recognizable, there exist integers $i_p \in E_1$ and $j_p \notin E_1$ such that $u_{[i_p,i_p+m_p)} = u_{[j_p,j_p+m_p)}$. The choice of m_p guarantees that $u_{[i_p,i_p+m_p)}$ has a natural p-cutting, say

$$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \dots, \sigma^p(u_{i'_p+k_p-1}), \beta_p\}.$$

Since $k_p \geq \dfrac{m_p}{\max_{a\in A}|\sigma^p(a)|} - 2 > k$, putting $\ell_p = |\alpha_p \sigma^p(u_{[i'_p,i'_p+k)})|$, we have a natural p-cutting $\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \dots, \sigma^p(u_{i'_p+k-1})\}$ of $u_{[i_p,i_p+\ell_p)}$. Since

$$\ell_p - |\alpha_p| \geq k \min_{a\in A}|\sigma^p(a)| \geq kC^{-1}\lambda^p \geq kC^{-2}\max_{a\in A}|\sigma^p(a)| \geq 3\max_{a\in A}|\sigma^p(a)|,$$

$u_{[j_p+|\alpha_p|,j_p+\ell_p)}$ has a natural p-cutting $\{\gamma_p, \sigma^p(u_{j'_p}), \sigma^p(u_{j'_p+1}), \dots, \sigma^p(u_{j'_p+h_p-1}), \gamma'_p\}$ with $\gamma_p \neq \Lambda$. □

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References

1. F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory Dynam. Systems **19** (1999), 953–993.
2. B. Host, *Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable*, Ergodic Theory Dynam. Systems **6** (1986), 529–540.
3. P. Kůrka, *Topological and Symbolic Dynamics*, Cours Spécialisés, vol. 11, Société Mathématique de France, Paris, 2003.
4. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1999.
5. B. Mossé, *Puissances de mots et reconnaissabilité des points fixes d'une substitution*, Theoret. Comput. Sci. **99** (1992), 327–334.
6. N. Pytheas-Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002, edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
7. M. Queffélec, *Substitution Dynamical Systems—Spectral Analysis*, Lecture Notes in Math., vol. 1294, Springer-Verlag, Berlin-New York, 1987.
8. ———, *Substitution Dynamical Systems—Spectral Analysis*, Second ed., Lecture Notes in Math., vol. 1294, Springer-Verlag, Berlin, 2010.

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