A REVIEW OF MOSSÉ'S RECOGNIZABILITY THEOREM

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Abstract. The goal of the present article is to supply B. Mossé's (unilateral) recognizability theorem with a proof easy to follow.

1. Introduction

Let A be a finite alphabet of at least two letters. Let σ be a primitive substitution over A which admits a fixed point u in $A^{\mathbb{Z}_+}$, where $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. For each $p \in \mathbb{N} := \{1, 2, 3, \dots\}$, set

$$E_p = \{0\} \cup \{|\sigma^p(u_{[0,n)})| : n \in \mathbb{N}\},$$

whose elements are called the *natural p-cutting points*. If $q \geq p$, then $E_q \subsetneq E_p$.

Definition 1 ([3, p. 530],[8, Definition V. 6.])**.** The substitution σ is said to be *recognizable* if there exists $L \in \mathbb{N}$ such that

$$u_{[i,i+L)} = u_{[j,j+L)}, i \in E_1 \Rightarrow j \in E_1.$$

The so-called Morse substitution: $a \mapsto ab, b \mapsto ba$ has a fixed point

$$u = abbabaabbaababbabaababbaabbabaab\dots$$

and then $E_1 = \{0, 2, 4, 6, \dots\}$. As is shown in [8, p. 109], the Morse substitution is recognizable with $L = 4$.

The recognizability is an important notion for primitive substitutions from viewpoints of associated subshifts. If the substitution σ is recognizable, then the unilateral subshift X_σ arising from σ has a Kakutani-Rohlin partition [2, 1] built on a *clopen* subset $\sigma(X_\sigma)$ of X_σ which pictures the substitution rule. It is also a significant consequence of the recognizability that $\sigma(X_\sigma)$ is open [8, Proposition VI. 3]. The property of $\sigma(X_\sigma)$ being open is actually equivalent to the recognizability [3]. In fact, [8, Proposition VI. 6] states that given a point $x \in X_\sigma$, the first return time of the point $\sigma(x)$ to the clopen subset $\sigma(X_\sigma)$ equals the length $|\sigma(x_0)|$ of the word $\sigma(x_0)$, and so the Kakutani-Rohlin partition is given by

$$\{T^k\sigma([a]) : 0 \leq k < |\sigma(a)|, a \in A\},$$

where T is the shift on X_σ, and $[a] = \{x = (x_i)_i \in X_\sigma | x_0 = a\}$. This leads to a fact that the first return map on $\sigma(X_\sigma)$ is a topological factor of X_σ, which shows a self-similarity of X_σ if σ is one-to-one on the alphabet A; see [8, Corollary VI. 8].

It is important to characterize a class of primitive substitutions with the recognizability. It is B. Mossé [6] that succeeded in the characterization:

Theorem 1 ([6, Théorème 3.1])**.** *If u is aperiodic, i.e. $T^n u \neq u$ for all $n \in \mathbb{N}$, then the following are equivalent.*

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(1) σ is not recognizable;
(2) for each $L \in \mathbb{N}$, there exist $i, j \in \mathbb{Z}_+$ such that
- *$\sigma(u_j)$ is a strict suffix of $\sigma(u_i)$;*
- *$\sigma(u_{i+k}) = \sigma(u_{j+k})$ for each integer k with $1 \leq k \leq L$.*

The author believes that B. Mossé's proof [6] for this theorem would be difficult to completely follow, in particular, Part (4) in p. 332. Moreover, no proofs can be found in recent textbooks [7, 4, 9], though a proof for the *bilateral* recognizability is written in [4, pp. 163-164]. This motivated the author to write the present article.

Everything which the reader will see in the present article is due to B. Mossé or other pioneers.

2. Preliminaries

Let A be a finite alphabet, i.e. a finite set, of at least two elements. An element of A is called a letter. Let A^+ denote the set of nonempty words over A. The empty word is denoted by Λ. Put $A^* = A^+ \cup \{\Lambda\}$. We say that a word $w \in A^*$ occurs in a word $v \in A^*$ if there exist $p, s \in A^*$ such that $v = pws$. We then write $w \prec v$. More specifically, w is said to occurs at the position $|p| + 1$ in v. The position is called an occurrence of w in v. Let $|v|_w$ denote the number of occurrences of w in v. The words p and s are called a prefix and suffix of v, respectively. We then write $p \prec_{\mathrm{p}} v$ and $s \prec_{\mathrm{s}} v$, respectively. If $|p| < |v|$ (resp. $|s| < |v|$), then p (resp. s) is called a strict prefix (resp. suffix) of v, and then we write $p \prec_{\mathrm{sp}} v$ (resp. $s \prec_{\mathrm{ss}} v$). A power of a word $w \in A^*$ is a word of the form

$$w = \underbrace{vv \dots v}_{n \text{ times}}$$

with some $v \in A^*$ and $n \in \mathbb{N}$, which is denoted by w^n.

A substitution over A is defined to be a map from A to A^+. The domain of σ is naturally extended to each of A^+ and $A^{\mathbb{Z}_+}$ by means of concatenation or juxtaposition of words. A recursive formula defines the powers σ^k for $k \in \mathbb{Z}_+$ on each of A^+ and $A^{\mathbb{Z}_+}$. We always assume that the substitution σ is primitive, i.e. there exists $k \in \mathbb{N}$ such that $a \prec \sigma^k(b)$ for any $a, b \in A$.

To prove Lemma 5, we will use Lemma 1 and Corollary 2 below. They concern Perron-Frobenius Theory for nonnegative, square matrices. Let M denote a matrix whose entries are all real. The matrix M is said to be nonnegative if the entries of M are nonnegative. A nonnegative, square matrix M is said to be primitive if there exists $n \in \mathbb{N}$ such that M^n is positive, i.e. the entries of M^n are positive. If a nonnegative, square matrix M is primitive, then it has such an eigenvalue $\lambda > 0$ that the absolute value of any other eigenvalue is less than λ. The eigenvalue λ is called the Perron eigenvalue of the matrix M. See for details [5, Sections 4.2 and 4.5]. The reader may refer to [5, Theorem 4.5.12] for a proof of the following lemma.

Lemma 1. *Let λ denote the Perron eigenvalue of a primitive matrix M. Let s denote the size of M. Then, there exist sets $\{c_{i,j} > 0 : 1 \leq i, j \leq s\}$ and $\{\rho_{i,j}(n) \in \mathbb{R} : 1 \leq i, j \leq s, n \in \mathbb{N}\}$ such that for all pair $(i, j) \in \{1, 2, \dots, s\}^2$ and $n \in \mathbb{N}$,*

$$\lim_{n \to \infty} \rho_{i,j}(n) = 0 \text{ and } (M^n)_{i,j} = \{c_{i,j} + \rho_{i,j}(n)\}\lambda^n.$$

The incidence matrix M_σ of the substitution σ is defined to be an $A \times A$ matrix whose (a,b)-entry equals $|\sigma(a)|_b$. Clearly, a substitution is primitive if and only if the incidence matrix associated with the substitution is primitive. Notice that every (a,b)-entry of ${M_\sigma}^n$ equals $|\sigma^n(a)|_b$.

Corollary 2. *Let λ denote the Perron eigenvalue of M_σ. Then, there exists an integer $C \geq 2$ such that for all $a \in A$ and $n \in \mathbb{N}$,*

$$C^{-1}\lambda^n \leq |\sigma^n(a)| \leq C\lambda^n.$$

Proof. Lemma 1 allows us to obtain $c_{a,b} > 0$, $\rho_{a,b}(n) \in \mathbb{R}$ such that for all $a, b \in A$,

$$\lim_{n\to\infty} \rho_{a,b}(n) = 0 \text{ and } |\sigma^n(a)|_b = \{c_{a,b} + \rho_{a,b}(n)\}\lambda^n.$$

Fix a letter $a \in A$. Since

$$\lambda^{-n}|\sigma^n(a)| > 0$$

for all $n \in \mathbb{N}$ and we have that

$$\lambda^{-n}|\sigma^n(a)| \to \sum_{b\in A} c_{a,b} \ (n \to \infty),$$

there exists $C_a \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$C_a^{-1} \leq \lambda^{-n}|\sigma^n(a)| \leq C_a.$$

Taking $C = \max_{a\in A} C_a + 1$, we obtain the desired conclusion. Notice that we require that $C \geq 2$. $\square$

Suppose that $u := u_0u_1u_2 \ldots \in A^{\mathbb{Z}_+}$ is a fixed point of σ, where $u_i \in A$ for each $i \in \mathbb{Z}_+$. Put

$$\mathcal{L}(u) = \{u_iu_{i+1}\ldots u_j | i, j \in \mathbb{Z}_+, i \leq j\} \cup \{\Lambda\}.$$

It follows from the primitivity of σ that u is uniformly recurrent, i.e. given a word $w \in \mathcal{L}(u)$, there exists $N \in \mathbb{N}$ such that

$$v \in \mathcal{L}(u), |v| = N \Rightarrow w \prec v.$$

We say that a word $w \in A^*$ occurs at a position $i \in \mathbb{Z}_+$ in u if

$$u_{[i,i+|w|)} := u_iu_{i+1}\ldots u_{i+|w|-1} = w.$$

3. Powers of words occurring in the fixed point

Definition 2. A word $v \in A^+$ is said to be *primitive* if it holds that

$$v = w^n, w \in A^+, n \in \mathbb{N} \Rightarrow (w = v, \text{ or equivalently } n = 1).$$

Lemma 3 ([6, Propriété 2.3])**.** *If $v \in A^+$ is primitive and $vwv \prec v^n$ for some $n \in \{2, 3, \ldots\}$, then w is a power of v.*

Proof. Assume that w is not any power of v. Since $vwv \prec v^n$, there exist $s_1, t_1 \in A^+$ such that

$$v = s_1t_1 = t_1s_1. \tag{3.1}$$

Since v is primitive, we have $|s_1| \neq |t_1|$. We may assume that $|s_1| < |t_1|$. Figure 1 shows the two ways (3.1) of decomposing v. Since v is primitive, it is necessary that $|s_1| \nmid |t_1|$. Find a word s_2 of length $(|t_1| \bmod |s_1|)$ such that

$$t_2 = {s_1}^{\lfloor |t_1|/|s_1| \rfloor} s_2.$$

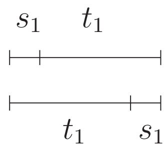

FIGURE 1. decompositions of v into s_1 and t_1

We then find a word $t_2 \in A^+$ such that $s_1 = t_2 s_2 = s_2 t_2$. We are again in the situation similar to Figure 1; replace v, s_1, t_1 with s_1, s_2, t_2, respectively. Continuing this procedure, we finally reach $n \in \mathbb{N}$ for which s_n is a letter. This forces that v is a power of the letter, which is a contradiction. $\square$

Lemma 4 ([6, Lemme 2.5]). *If there exist $N, p \in \mathbb{N}$ and a primitive word $v \in A^+$ such that*

(1) $\sigma^p(u_i u_{i+1}) \prec v^N$ for all $i \in \mathbb{Z}_+$;
(2) $2|v| \leq \min_{a\in A} |\sigma^p(a)|$,

then u is periodic.

Proof. It follows from (1) that each $\sigma^p(u_i) \prec v^N$, so that for each $i \in \mathbb{Z}_+$, there exist $s_i \prec_{\rm ss} v$, $t_i \prec_{\rm sp} v$ and $n_i \in \mathbb{Z}_+$ such that

$$\sigma^p(u_i) = s_i v^{n_i} t_i.$$

If $n_i = 0$ for some $i \in \mathbb{Z}_+$, then $|\sigma^p(u_i)| < 2|v|$, which contradicts (2). Hence, $n_i \geq 1$ for all $i \in \mathbb{Z}_+$. This guarantees that for all $i \in \mathbb{Z}_+$,

$$v t_i s_{i+1} v \prec s_i v^{n_i} t_i s_{i+1} v^{n_{i+1}} t_{i+1} = \sigma^p(u_i u_{i+1}) \prec v^N.$$

It follows from Lemma 3 that each $t_i s_{i+1}$ is a power of v. We obtain finally that

$$u = \sigma^p(u) = s_0 v^{n_0} t_0 s_1 v^{n_1} t_1 s_2 v^{n_2} t_2 \cdots = s_0 v^\infty,$$

which is periodic, because $s_0 \prec_{\rm ss} v$. This completes the proof. $\square$

In order to prove Theorem 1, we will use the following lemma, which itself is actually interesting.

Lemma 5 ([6, Théorème 2.4]). *If u is aperiodic, then there exists $N \in \mathbb{N}$ such that*

$$\sup\{n \in \mathbb{N} : w^n \in \mathcal{L}(u), w \in A^+\} < N.$$

It is clearly necessary that $N > 1$.

Proof. Suppose that some power w^n of a primitive word $w \in A^+$ occurs in u. For $p \in \mathbb{Z}_+$, put

$$\ell_p = \frac{1}{2} \min_{a\in A} |\sigma^p(a)|.$$

Since $\ell_p \uparrow \infty$, there uniquely exists $p \in \mathbb{N}$ such that

$$\ell_{p-1} \leq |w| < \ell_p.$$

Since u is uniformly recurrent, there exists $g \in \mathbb{N}$ so that for all $i, j \in \mathbb{Z}$,

$$u_i u_{i+1} \prec u_{[j,j+g)}.$$

It follows that for all $i, j \in \mathbb{Z}$,

$$\sigma^p(u_i u_{i+1}) \prec \sigma^p(u_{[j,j+g)}).$$

If the length of a word $w' \in \mathcal{L}(u)$ equals

$$(g+2)\max_{a\in A}|\sigma^p(a)|,$$

then $\sigma^p(u_{[j,j+g)}) \prec w'$ for some $j \in \mathbb{Z}_+$, so that $\sigma^p(u_i u_{i+1}) \prec w'$ for all $i \in \mathbb{Z}_+$. Since $2|w| < \min_{a\in A}|\sigma^p(a)|$ and u is aperiodic, Lemma 4 forces that

$$|w^n| = n|w| < (g+2)\max_{a\in A}|\sigma^p(a)|.$$

Then

$$n < \frac{(g+2)\max_{a\in A}|\sigma^p(a)|}{\ell_{p-1}} = 2(g+2)\frac{\max_{a\in A}|\sigma^p(a)|}{\min_{a\in A}|\sigma^{p-1}(a)|} \le 2(g+2)\lambda C^2,$$

where λ and C are as in Corollary 2. This completes the proof. □

4. A proof of Theorem 1

Definition 3. (1) Let $i \in \mathbb{Z}_+$ and $p, \ell \in \mathbb{N}$. A finite sequence

$$\{\alpha, \sigma^p(u_{i'}), \sigma^p(u_{i'+1}), \dots, \sigma^p(u_{i'+k-1}), \beta\},$$

where $i' \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, is called a *natural p-cutting* of $u_{[i,i+\ell)}$ if
(a) $u_{[i,i+\ell)} = \alpha\sigma^p(u_{i'})\sigma^p(u_{i'+1})\dots\sigma^p(u_{i'+k-1})\beta$;
(b) $\alpha \prec_{\mathrm{s}} \sigma^p(u_{i'-1})$;
(c) $\beta \prec_{\mathrm{p}} \sigma^p(u_{i'+k})$;
(d) $i + |\alpha| = |\sigma^p(u_{[0,i')})|$.

(2) Suppose that a word w occurs at positions i and j in u. The word w is said to have the *same p-cutting* at the positions i and j if

$$(E_p \cap [i, i+|w|-1]) + (j-i) = E_p \cap [j, j+|w|-1].$$

Remark 4. We do not exclude the possibility that $\alpha = \Lambda$ or $\beta = \Lambda$. Not every $u_{[i,i+\ell)}$ has a natural p-cutting, because we require $k \ge 1$ in Definition 3 (1). It is not necessary that a natural p-cutting is uniquely determined for given i and ℓ in Definition 3 (1).

We now proceed to our proof of Theorem 1.

Proof of Theorem 1. (1) ⇒ (2): Fix an integer k with

$$k \ge C^2(C^2N+2) > 4C^2,$$

where C (resp. N) is as in Corollary 2 (resp. Lemma 5). Assume that σ is not recognizable.

Step 1. For each $p \in \mathbb{N}$, there exist $i_p \in E_1$, $j_p \in \mathbb{Z}_+ \setminus E_1$, $i'_p, j'_p \in \mathbb{Z}_+$, $h_p, \ell_p \in \mathbb{N}$ and $\alpha_p, \gamma'_p \in A^*$, $\gamma_p \in A^+$ such that

- $u_{[i_p,i_p+\ell_p)} = u_{[j_p,j_p+\ell_p)}$;
- $$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \dots, \sigma^p(u_{i'_p+k-1})\}$$
  is a natural p-cutting of $u_{[i_p,i_p+\ell_p)}$;
- $$\{\gamma_p, \sigma^p(u_{j'_p}), \sigma^p(u_{j'_p+1}), \dots, \sigma^p(u_{j'_p+h_p-1}), \gamma'_p\}$$
  is a natural p-cutting of $u_{[j_p+|\alpha_p|,j_p+\ell_p)}$.

This fact is not difficult to verify; see Lemma 6 below the present proof. The above properties allows us to define

$$m_p = \min\{m \in \mathbb{N} : \alpha_p\gamma_p \prec_{\mathrm{s}} \sigma^p(u_{j'_p-m})\sigma^p(u_{j'_p-m+1})\dots\sigma^p(u_{j'_p-1})\}.$$

Since $\alpha_p \prec_{\mathrm{s}} \sigma^p(u_{i'_p-1})$ and $\gamma_p \prec_{\mathrm{s}} \sigma^p(u_{j'_p-1})$, it follows that for all $p \in \mathbb{N}$,

$$m_p \leq \frac{|\alpha_p\gamma_p|}{\min_{a\in A}|\sigma^p(a)|} \leq \frac{2\max_{a\in A}|\sigma^p(a)|}{\min_{a\in A}|\sigma^p(a)|} \leq 2C^2,$$

so that $m := \max_{p\in\mathbb{N}} m_p$ exists and is independent of the choice of k. We know that for all $p \in \mathbb{N}$,

$$\alpha_p\gamma_p \prec_{\mathrm{s}} \sigma^p(u_{j'_p-m})\sigma^p(u_{j'_p-m+1})\dots\sigma^p(u_{j'_p-1}).$$

Step 2. Since

$$\gamma_p\sigma^p(u_{j'_p})\sigma^p(u_{j'_p+1})\dots\sigma^p(u_{j'_p+h_p-1})\gamma'_p = \sigma^p(u_{i'_p})\sigma^p(u_{i'_p+1})\dots\sigma^p(u_{i'_p+k-1}), \tag{4.1}$$

it follows that

$$h_p \min_{a\in A}|\sigma^p(a)| \leq k \max_{a\in A}|\sigma^p(a)|,$$

so that $h_p \leq kC^2$. Hence, $\{h_p\}_p$ is bounded. Recall that k is now fixed. Hence, the set

$$\{(h_p, u_{[i'_p-1,i'_p+k-1]}, u_{[j'_p-m,j'_p+h_p]}) \in \mathbb{N}\times A^+ \times A^+ : p \in \mathbb{N}\}$$

is finite. We can find an infinite set $I \subset \mathbb{N}$, $h \in \mathbb{N}$ and words

$$a_{-1}a_0\dots a_{k-1}, b_{-m}b_{-m+1}\dots b_h \in \mathcal{L}(u)\ (a_i, b_j \in A)$$

so that for all $p \in I$,

$$h_p = h; \tag{4.2}$$

$$u_{[i'_p-1,i'_p+k-1]} = a_{-1}a_0\dots a_{k-1}; \tag{4.3}$$

$$u_{[j'_p-m,j'_p+h_p]} = b_{-m}b_{-m+1}\dots b_h. \tag{4.4}$$

Step 3. It follows from (4.1)-(4.4) that for any $p, q \in I$ with $p < q$,

$$\gamma_q\sigma^q(b_0b_1\dots b_{h-1})\gamma'_q = \sigma^{q-p}(\gamma_p)\sigma^q(b_0b_1\dots b_{h-1})\sigma^{q-p}(\gamma'_p). \tag{4.5}$$

Assume that $\sigma^{q-p}(\gamma_p) \neq \gamma_q$ for some $p, q \in I$ with $p < q$. Consider the case $|\gamma_q| > |\sigma^{q-p}(\gamma_p)|$. Equation (4.5) allows us to find a word $v \in A^+$ such that

- $\gamma_q = \sigma^{q-p}(\gamma_p)v \prec_{\mathrm{s}} \sigma^q(b_{-1})$;
- $v^{N'} \prec_{\mathrm{p}} \sigma^q(b_0b_1\dots b_{h-1})$ and $v^{N'+1} \not\prec_{\mathrm{p}} \sigma^q(b_0b_1\dots b_{h-1})$ for some $N' \in \mathbb{N}$.

Deducing from (4.1)-(4.4) that

$$(h+2)\max_{a\in A}|\sigma^q(a)| \geq k \min_{a\in A}|\sigma^q(a)|.$$

This together with the fact that $v \prec_{\mathrm{s}} \sigma^q(b_{-1})$ implies that

$$N' \geq \frac{h\min_{a\in A}|\sigma^q(a)|}{\max_{a\in A}|\sigma^q(a)|} \geq (kC^{-2}-2)C^{-2} \geq N,$$

which contradicts Lemma 5. Also, in the other case, we reach the same contradiction. It follows therefore that $p, q \in I, q > p \Rightarrow \gamma_q = \sigma^{q-p}(\gamma_p)$.

Step 4. Since I is infinite, σ is primitive and $\gamma_p \neq \Lambda$, we can fix $p, q \in I$ with $q > p$ so that $|\sigma^{q-p-1}(\gamma_p)| > L$.

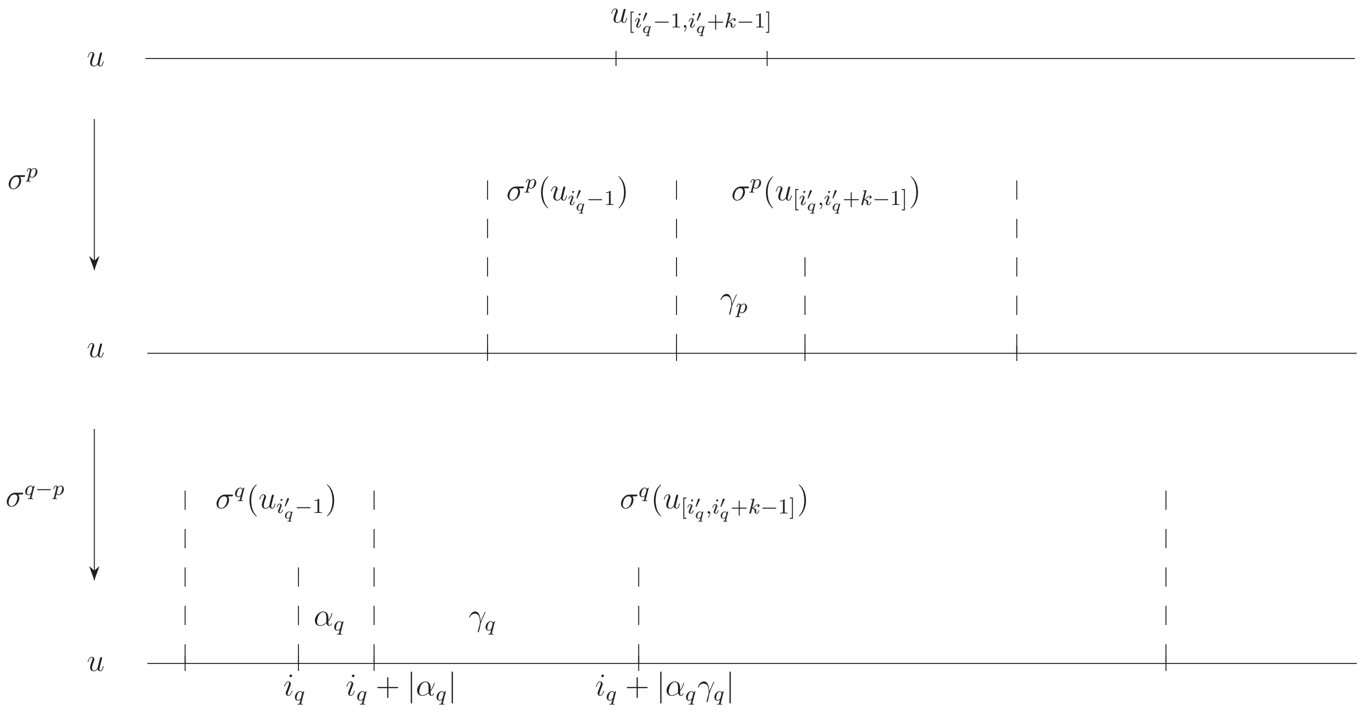

FIGURE 2

Observe how powers of σ map $u_{[i'_q,i'_q+k-1]}$ to $\sigma^q(u_{[i'_q,i'_q+k-1]})$ via $\sigma^{q-p}(u_{[i'_q,i'_q+k-1]})$; see Figure 2. Since

$$\gamma_p \prec_{\mathrm{p}} \sigma^p(u_{[i'_p,i'_p+k-1]}) = \sigma^p(u_{[i'_q,i'_q+k-1]});$$
$$\gamma_q \prec_{\mathrm{p}} \sigma^q(u_{[i'_q,i'_q+k-1]});$$
$$\sigma^{q-p}(\gamma_p) = \gamma_q,$$

it follows that

(i) $i_q + |\alpha_q| \in E_q \subset E_1$;
(ii) $i_q + |\alpha_q\gamma_q| \in E_{q-p} \subset E_1$;
(iii) $\sigma^{q-p-1}(\gamma_p)$ occurs at the position $i'' := |\sigma^{q-p-1}(u_{[0,|\sigma^p(u_{[0,i'_q)})|)})|$ in u;
(iv)
$$\{\sigma(u_{i''}), \sigma(u_{i''+1}), \ldots, \sigma(u_{i''+|\sigma^{q-p-1}(\gamma_p)|-1})\}$$
is a natural 1-cutting of $u_{[i_q+|\alpha_q|,i_q+|\alpha_q\gamma_q|)} = \gamma_q$;

Then, observe how powers of σ map $u_{[j'_q-m,j'_q+h]}$ to $\sigma^q(u_{[j'_q-m,j'_q+h]})$ via $\sigma^{q-p}(u_{[j'_q-m,j'_q+h]})$; see Figure 3. Then,

(v) $j_q + |\alpha_q| \in E_{q-p} \subset E_1$;
(vi) $j_q + |\alpha_q\gamma_q| \in E_{q-p} \subset E_1$;
(vii) $\sigma^{q-p-1}(\gamma_p)$ occurs at the position $j'' := |\sigma^{q-p-1}(u_{[0,|\sigma^p(u_{[0,j'_q)})|-|\gamma_p|)})|$ in u;
(viii)
$$\{\sigma(u_{j''}), \sigma(u_{j''+1}), \ldots, \sigma(u_{j''+|\sigma^{q-p-1}(\gamma_p)|-1})\}$$
is a natural 1-cutting of $u_{[j_q+|\alpha_q|,j_q+|\alpha_q\gamma_q|)} = \gamma_q$.

It follows from (iii), (iv), (vii) and (viii) that γ_q has the same 1-cutting at the positions $i_q + |\alpha_q|$ and $j_q + |\alpha_q|$ in such a way that (i), (ii), (vi) and (v) hold. In

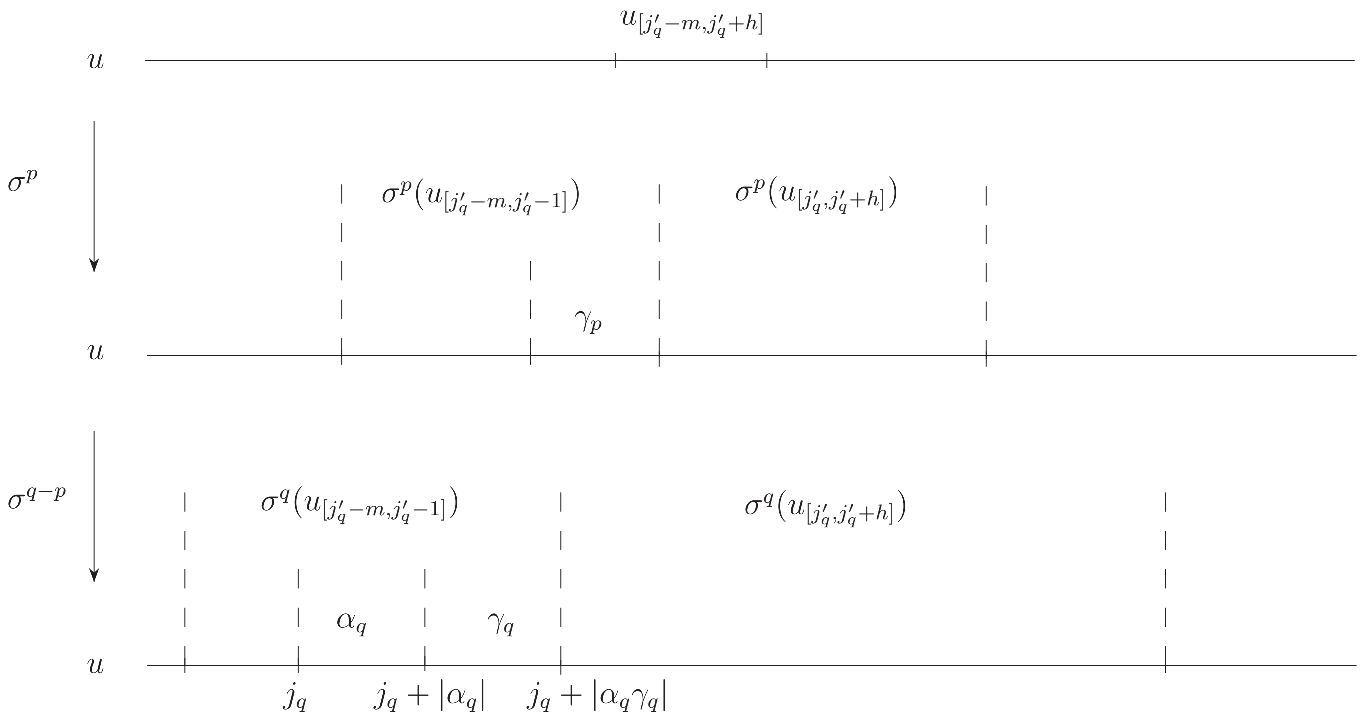

FIGURE 3

view of this together with the fact that $i_q \in E_1$ and $j_q \notin E_1$, we can find sequences:

$$i_q \le s_{-n} < s_{-n+1} < \cdots < s_{-1} < s_0 = i_q + |\alpha_q|;$$
$$t_{-n} < t_{-n+1} < \cdots < t_{-1} < t_0 = j_q + |\alpha_q|$$

of consecutive, natural 1-cutting points, i.e. $s_{i-1} < s < s_i \Rightarrow s \notin E_1$, such that

- $s_i - s_{i-1} = t_i - t_{i-1}$ for all integers i with $-n+1 < i \le 0$;
- $s_{-n+1} - s_{-n} \neq t_{-n+1} - t_{-n}$.

It may happen that $t_{-n} < j_q$. This completes the proof, because the converse implication (2) $\Rightarrow$ (1) is obvious. $\square$

Lemma 6. *Let k be an arbitrary integer with $k > 4C^2$, where C is a constant as in Corollary 2. Assume that the substitution σ is not recognizable. Then, for every $p \in \mathbb{N}$, there exist*

$$i_p \in E_1, j_p \in \mathbb{Z}_+ \setminus E_1, i'_p, j'_p \in \mathbb{Z}_+, h_p, \ell_p \in \mathbb{N}, \alpha_p, \gamma'_p \in A^* \text{ and } \gamma_p \in A^+$$

such that

- $u_{[i_p,i_p+\ell_p)} = u_{[j_p,j_p+\ell_p)}$;
- $$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \ldots, \sigma^p(u_{i'_p+k-1})\}$$
  is a natural p-cutting of $u_{[i_p,i_p+\ell_p)}$;
- $$\{\gamma_p, \sigma^p(u_{j'_p}), \sigma^p(u_{j'_p+1}), \ldots, \sigma^p(u_{j'_p+h_p-1}), \gamma'_p\}$$
  is a natural p-cutting of $u_{[j_p+|\alpha_p|,j_p+\ell_p)}$.

Proof. Fix an arbitrary integer ℓ'_p with

$$\ell'_p > (k+2)\max_{a\in A}|\sigma^p(a)|. \tag{4.6}$$

In view of the assumption that σ is not recognizable, there exist $i_p \in E_1$ and $j_p \in \mathbb{Z}_+ \setminus E_1$ such that

$$u_{[i_p, i_p+\ell'_p)} = u_{[j_p, j_p+\ell'_p)}.$$

Condition 4.6 guarantees the existence of a natural p-cutting

$$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \dots, \sigma^p(u_{i'_p+k_p-1}), \alpha'_p\} \tag{4.7}$$

of $u_{[i_p, i_p+\ell'_p)}$. Since

$$k_p \geq \frac{\ell'_p}{\max_{a\in A}|\sigma^p(a)|} - 2 > k,$$

from (4.7), we can choose a natural p-cutting

$$\{\alpha_p, \sigma^p(u_{i'_p}), \sigma^p(u_{i'_p+1}), \dots, \sigma^p(u_{i'_p+k-1})\}$$

of $u_{[i_p, i_p+\ell_p)}$, where

$$\ell_p = |\alpha_p \sigma^p(u_{[i'_p, i'_p+k-1]})|.$$

Since

$$\begin{aligned}
\ell_p - |\alpha_p| &\geq \left(k\frac{\min_{a\in A}|\sigma^p(a)|}{\max_{a\in A}|\sigma^p(a)|} - 1\right)\max_{a\in A}|\sigma^p(a)| \\
&\geq (kC^{-2} - 1)\max_{a\in A}|\sigma^p(a)| \\
&> 3\max_{a\in A}|\sigma^p(a)|,
\end{aligned}$$

we can choose a natural p-cutting

$$\{\gamma_p, \sigma^p(u_{j'_p}), \sigma^p(u_{j'_p+1}), \dots, \sigma^p(u_{j'_p+h_p-1}), \gamma'_p\}.$$

of $u_{[j_p+|\alpha_p|, j_p+\ell_p)}$ so that $\gamma_p \neq \Lambda$. $\square$

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