ウェーブレット

No. 1

Date . .

$e^{-t^2/2}$（ガウス関数）の2段階微分

$$(e^{-t^2/2})' = e^{-t^2/2}\cdot(-t^2/2)' = e^{-t^2/2}\cdot(-t) = -t\cdot e^{-t^2/2}$$

$$(-t\cdot e^{-t^2/2})' = (-t)'\cdot e^{-t^2/2} + (-t)\cdot(e^{-t^2/2})' = -e^{-t^2/2} + (-t)\cdot\{(-t)\cdot e^{-t^2/2}\} = (t^2-1)\cdot e^{-t^2/2}$$

よって、メキシカンハットウェーブレット

$$\psi(t) = -(t^2-1)\cdot e^{-t^2/2} = (1-t^2)\cdot e^{-t^2/2}$$

$$\hat{\psi}(\omega) = \int_{-\infty}^{\infty}\psi(t)e^{-i\omega t}dt$$

$$e^{-t^2/2} \xrightarrow[\text{フーリエ変換}]{\mathcal{F}} \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}$$

$$\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\psi}(\omega)e^{i\omega t}d\omega$$

$$\psi'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)\hat{\psi}(\omega)e^{i\omega t}d\omega$$

$$\psi''(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boxed{(i\omega)^2\hat{\psi}(\omega)}e^{i\omega t}d\omega$$

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$p=2$ $\quad ||f||_2 = \sqrt{\int_{-\infty}^{\infty}|f(t)|^2dt} < +\infty$ $\quad L^2$ ノルム

内積 $\quad \langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt$, $\quad |\langle f,g\rangle| \leq ||f||_2\cdot||g||_2$

2乗エネルギー

$$\langle f, f\rangle = \int_{-\infty}^{\infty}|f(t)|^2dt$$

$$\mathcal{F}(f) = \hat{f}(\omega) = \int f(t)e^{-i\omega t}dt = \langle f, \cancel{e^{-i\omega t}}\rangle$$

$$f(t) = \frac{1}{2\pi}\int\hat{f}(\omega)e^{i\omega t}dt$$

$t-x=s$

$T_x f(t) = f(t-x)$

平行移動

$$\langle T_x f, g\rangle = \int_{-\infty}^{\infty} f(t-x)\overline{g(t)}\,dt = \int_{-\infty}^{\infty} f(s)\overline{g(x+s)}\,ds = \int_{-\infty}^{\infty} f(s)\overline{T_{-x}g(s)}\,ds = \langle f, T_{-x}g\rangle$$

ジョイント //

$\langle f, T_x^* g\rangle$

$T_x^* = T_{-x}$

No.

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モジュレーション

$$M_\xi f(t) = e^{it\xi} f(t)$$

$$\begin{aligned}
\langle M_\xi f, g \rangle &= \int_{-\infty}^{\infty} e^{i\omega\xi} f(t)\, \overline{g(t)}\, dt \\
&= \int_{-\infty}^{\infty} f(t)\, \overline{e^{-it\xi} g(t)}\, dt \\
&= \langle f, \underbrace{e^{-it\xi} g}_{M_{-\xi}\, g} \rangle
\end{aligned}$$

$$M_\xi^* = M_{-\xi}$$

ダイレーション

$$\rho > 0, \quad D_\rho f(t) = \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right)$$

$$\begin{aligned}
\langle D_\rho f, g \rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right) \overline{g(t)}\, dt \qquad \frac{t}{\rho} = s \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\rho}} f(s)\, \overline{g(\rho s)}\, \rho\, ds \qquad \frac{dt}{\rho} = ds
\end{aligned}$$

$$\sqrt{\rho}$$

$$= \int_{-\infty}^{\infty} f(s)\, \underbrace{\overline{\sqrt{\rho}\, g(\rho s)}}_{D_{1/\rho}\, g}\, ds = \langle f, D_{1/\rho}\, g \rangle \qquad D_\rho^* = D_{1/\rho}$$

ユニタリ

$$\|T_x f\|_2^2 = \|f\|_2^2$$

$$\begin{aligned}
(\text{左辺}) &= \int_{-\infty}^{\infty} |T_x f|^2\, dt \\
&= \int_{-\infty}^{\infty} |f(t-x)|^2\, dt \\
&= \int_{-\infty}^{\infty} |f(s)|^2\, ds \\
&= \|f\|_2^2
\end{aligned}$$

$$\|M_\xi f\|_2^2 = \|f\|_2^2$$

$$\begin{aligned}
(\text{左辺}) &= \int_{-\infty}^{\infty} |M_\xi f|^2\, dt \\
&= \int_{-\infty}^{\infty} |e^{it\xi} f(t)|^2\, dt \\
&= \int_{-\infty}^{\infty} |f(t)|^2\, dt \\
&= \|f\|_2^2
\end{aligned}$$

$$\|D_\rho f\|_2^2 = \|f\|^2$$

$$\begin{aligned}
(\text{左辺}) &= \int_{-\infty}^{\infty} |D_\rho f|^2\, dt \\
&= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{\rho}} f(s) \right|^2 \rho\, ds \\
&= \int_{-\infty}^{\infty} |f(s)|^2 \left|\frac{1}{\rho}\right| \cdot \rho\, ds \\
&= \int_{-\infty}^{\infty} |f(s)|^2\, ds \\
&= \|f\|_2^2
\end{aligned}$$

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Lemma 2.1

$$T_x M_\xi f(t) = T_x\left(e^{it\xi} f(t)\right) = e^{i(t-x)\xi} f(t-x)$$
$$e^{-i\xi x} M_\xi T_x f(t) = e^{-i\xi x} M_\xi f(t-x)$$
$$= e^{-i\xi x} \cdot e^{it\xi} f(t-x) = e^{i(t-x)\xi} f(t-x)$$

$$M_\xi T_x f(t) = M_\xi f(t-x) = e^{it\xi} f(t-x)$$
$$e^{i\xi x} T_x M_\xi f(t) = e^{i\xi x} T_x e^{it\xi} f(t)$$
$$= e^{i\xi x} e^{i(t-x)\xi} f(t-x) = e^{it\xi} f(t-x)$$

$$T_x D_\rho f(t) = T_x \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right) = \frac{1}{\sqrt{\rho}} f\left(\frac{t-x}{\rho}\right)$$
$$D_\rho T_{x/\rho} f(t) = D_\rho f\left(t - \frac{x}{\rho}\right) = \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho} - \frac{x}{\rho}\right) = \frac{1}{\sqrt{\rho}} f\left(\frac{t-x}{\rho}\right)$$

$$D_\rho T_x f(t) = D_\rho f(t-x) = \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho} - x\right)$$
$$T_{\rho x} D_\rho f(t) = T_{\rho x} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right) = \frac{1}{\sqrt{\rho}} f\left(\frac{t-\rho x}{\rho}\right) = \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho} - x\right)$$

$$M_\xi D_\rho f(t) = M_\xi \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right) = e^{it\xi} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right)$$
$$D_\rho M_{\rho\xi} f(t) = D_\rho e^{it\rho\xi} f(t) = \frac{1}{\sqrt{\rho}} e^{i\frac{t}{\rho}\rho\xi} f\left(\frac{t}{\rho}\right) = e^{it\xi} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right)$$

$$D_\rho M_\xi f(t) = D_\rho e^{it\xi} f(t) = \frac{1}{\sqrt{\rho}} e^{i\frac{t}{\rho}\xi} f\left(\frac{t}{\rho}\right) = e^{it\xi/\rho} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right)$$
$$M_{\xi/\rho} D_\rho f(t) = M_{\xi/\rho} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right) = e^{it\xi/\rho} \frac{1}{\sqrt{\rho}} f\left(\frac{t}{\rho}\right)$$

Lemma 2.2

$$\mathcal{F} T_x f = \int f(\underbrace{t-x}_{s}) e^{-i\underbrace{t}_{s+x}\omega} dt$$
$$= \int f(s) e^{-is\omega} e^{-ix\omega} ds$$
$$= e^{-ix\omega} \int f(s) e^{-i\omega s} ds$$
$$M_{-x} \mathcal{F} f = M_{-x} \hat{f}(\omega)$$
$$= M_{-x} \int f(t) e^{-i\omega t} dt$$
$$= e^{-ix\omega} \int f(t) e^{-i\omega t} dt$$

$$\mathcal{F} M_\xi f = \mathcal{F} e^{it\xi} f(t)$$
$$= \int e^{it\xi} f(t) e^{-i\omega t} dt$$
$$= \int f(t) e^{-it(\omega-\xi)} dt$$
$$T_\xi \mathcal{F} f = T_\xi \hat{f}(\omega)$$
$$= T_\xi \int f(t) e^{-i\omega t} dt \qquad t - \xi = s$$
$$= \int f(t-\xi) e^{-i(t-\xi)\omega} dt$$
$$= \int f(s) e^{-i\omega s} ds$$

$$\mathcal{F} D_\rho = D_{1/\rho} \mathcal{F}$$

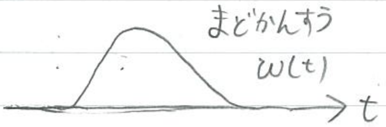
$$\begin{aligned}
T_x (\mathcal{F}^{-1} g(\omega))(t) &= T_x \frac{1}{2\pi} \int g(\omega) e^{it\omega} d\omega \\
&= \frac{1}{2\pi} \int g(\omega) e^{i(t-x)\omega} d\omega \\
&= \frac{1}{2\pi} \int g(\omega) e^{-ix\omega} e^{it\omega} d\omega \\
&= \mathcal{F}^{-1} (g(\omega) e^{-ix\omega}) \\
&= \mathcal{F}^{-1} M_{-x} g(\omega)
\end{aligned}$$

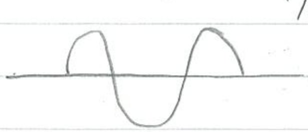
$$\begin{aligned}
M_\xi (\mathcal{F}^{-1} g(\omega))(t) &= M_\xi \frac{1}{2\pi} \int g(\omega) e^{it\omega} d\omega \\
&= \frac{1}{2\pi} \int e^{it\xi} g(\omega) e^{it\omega} d\omega \\
&= \frac{1}{2\pi} \int g(\omega) e^{it(\omega+\xi)} d\omega \\
&= \frac{1}{2\pi} \int g(\omega-\xi) e^{it\omega} d\omega \\
&= \mathcal{F}^{-1} (g(\omega-\xi)) = \mathcal{F}^{-1} T_\xi g(\omega)
\end{aligned}$$

$$\begin{aligned}
D_\rho (\mathcal{F}^{-1} g(\omega))(t) &= D_\rho \frac{1}{2\pi} \int g(\omega) e^{it\omega} d\omega \\
&= \frac{1}{2\pi} \int \frac{1}{\sqrt{\rho}} g(\omega) e^{it\omega/\rho} d\omega \\
&= \frac{1}{2\pi} \int \frac{1}{\sqrt{\rho}} g(\rho s) e^{its} \cdot \rho \, ds \\
&= \frac{1}{2\pi} \int \sqrt{\rho}\, g(\rho s) e^{its} ds \\
&= \mathcal{F}^{-1} (\sqrt{\rho}\, g(\rho s)) = \mathcal{F}^{-1} D_{1/\rho} g
\end{aligned}$$

$\frac{\omega}{\rho} = s$

$\frac{d\omega}{\rho} = ds$





ウェーブレット

$\psi(t)$ $\quad \int \psi(t) dt = 0$

$\langle f, M_\xi T_x \omega \rangle$

まどフーリエ

短時間フーリエ変換

$= \int f(t) \overline{e^{it\xi} \omega(t-x)} dt$

$= \int f(t) \overline{\omega(t-x)} e^{-it\xi} dt$

$\langle f, T_b D_a \psi \rangle$

連続ウェーブレット変換

$T_b D_a \psi = T_b \frac{1}{\sqrt{a}} \psi\left(\frac{t}{a}\right)$

$= \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$