3月9日 (2011)

2乗エネルギー　　$n=1$.

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$$

↑ 複素共役

$$\|f\| = \sqrt{\langle f, f \rangle}$$

ノルム

$$T_b f(x) = f(x-b), \quad D_a = \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right)$$

$$M_\omega f(x) = e^{ix\omega} f(x)$$

$$\begin{aligned} T_b M_\omega f &= T_b\, e^{ix\omega} f(x) \\ &= e^{i(x-b)\omega} f(x-b) \\ &= \boxed{e^{ix\omega}}\, e^{-ib\omega} f(x-b) \\ &= e^{-ib\omega} \boxed{M_\omega} T_b f. \end{aligned}$$

$$\hat{f}(\xi) = \int f(x)\, e^{-ix\xi}\,dx$$

$$\mathcal{F}\underbrace{T_b f}_{f(x-b)} = \overbrace{M_{-b}}^{e^{-ib\xi}}\,\overbrace{\mathcal{F}f}^{\hat{f}(\xi)}$$

$x-b=s$

$$\begin{aligned} (\text{左辺}) &= \int f(\underbrace{x-b}_{s})\, e^{-ix\xi}\,dx \\ &= \int f(s)\, e^{-i(s+b)\xi}\,ds \\ &= e^{-ib\xi} \int f(s)\, e^{is\xi}\,ds = e^{-ib\xi}\,\hat{f}(\xi) \end{aligned}$$

iii) $T_b D_a f = T_b \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right)$

$$= \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right)$$

$$= \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right) f\left(-\frac{b}{a}\right)$$

$$= D_a T_{\frac{b}{a}} f \quad //$$

(ix) $\mathcal{F} D_a f = \int \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right) e^{-ix\xi} dx$

$$= \frac{1}{\sqrt{a}} \int f(s)\, e^{-ias\xi} \cdot a\, ds$$

$$= \frac{a}{\sqrt{a}} \int f(s)\, e^{-i(a\xi)s} ds$$

$$= \sqrt{a}\, \hat{f}(a\xi)$$

$$= D_{\frac{1}{a}} \mathcal{F} f.$$

$\frac{x}{a} = s$

$x = as$

$dx = a\,ds.$

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ix\xi} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2+2ix\xi}{2}} dx$$

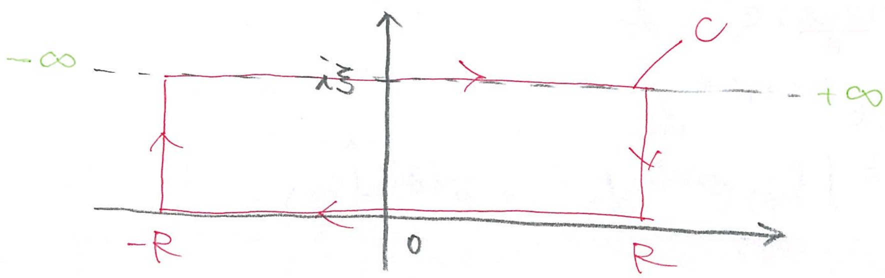
$-\{(x+i\xi)^2 - (i\xi)^2\}$

$-\{(x+i\xi)^2 + (\xi)^2\}$

$$= \int_{-\infty}^{\infty} e^{-\frac{(x+i\xi)^2}{2} - \frac{\xi^2}{2}} dx$$

$$= e^{-\frac{\xi^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x+i\xi)^2}{2}} dx$$

$z = x + i\xi$ とすると、



(正則より)

$$\therefore \quad 0 = \int_C e^{-\frac{z^2}{2}} dz$$

$$= \int_{-R+i\xi}^{R+i\xi} + \int_{R+i\xi}^{R} + \int_{R}^{-R} + \int_{-R}^{-R+i\xi}$$

$R \to \infty$

$$0 = \int_{-\infty+i\xi}^{\infty+i\xi} + \int_{\infty}^{-\infty}$$

$$\left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$

$$\begin{cases} x = r\cos\theta & (0 \leqq r \leqq \infty) \\ y = r\sin\theta & (0 \leqq \theta \leqq 2\pi) \end{cases}$$

$$dxdy = r\,dr\,d\theta$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} \cdot r \, dr\, d\theta.$$

$\frac{1}{e^{\infty}} = 0$.

$$= \int_0^{\infty} \left[r e^{-\frac{r^2}{2}} \cdot \theta \right]_0^{2\pi} dr$$

$$= 2\pi \int_0^{\infty} r e^{-\frac{r^2}{2}} dr$$

$e^{x^2} = 2x e^{x^2}$

$$= 2\pi \left[-e^{-\frac{r^2}{2}} \right]_0^{\infty}$$

$e^{-\frac{1}{2}r^2} = -r e^{-\frac{r^2}{2}}$

$$= 2\pi(-0+1)$$

$-\frac{1}{r}$

$$= 2\pi \,/\!/$$

$$\therefore\ \mathcal{F}(e^{-\frac{x^2}{2}})(\xi) = \sqrt{2\pi}\cdot e^{-\frac{\xi^2}{2}}$$

$$\mathcal{F}^{-1}(g(\xi))(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)\, e^{ix\xi}\, d\xi$$

$$\left(\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}}\cdot e^{ix\xi}\, d\xi\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{a^2}{2}} e^{ixa}\, da\right)\left(\int_{-\infty}^{\infty} e^{-\frac{b^2}{2}} e^{ixb}\, db\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{a^2+b^2}{2}}\, e^{i(a+b)x}\, da\, db$$

$$\begin{cases} z = a+b \\ a = r\cos\theta & (0 \leqq r \leqq \infty) \\ b = r\sin\theta & (0 \leqq \theta \leqq 2\pi) \end{cases} \qquad da\,db = r\,dr\,d\theta$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}}\, e^{ie^{i\theta}x}\cdot r\,dr\,d\theta \quad (途中)$$

$$\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} e^{ix\xi}\, d\xi = \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} e^{-ix\xi}\, d\xi$$

$$= \sqrt{2\pi}\, e^{-\frac{x^2}{2}}$$

$f*g\,(x) = \int f(y)\,g(x-y)\,dy$ ：合成積（たたみ込み）

$$\mathcal{F}(f*g)(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$$

$$\mathcal{F}(f*g)(\xi) = \int\int f(y)\,g(x-y)\,dy\, e^{-ix\xi}\, dx$$

$$\begin{cases} s = x-y \\ t = y \end{cases} \qquad ds\,dt = dy\,dx$$

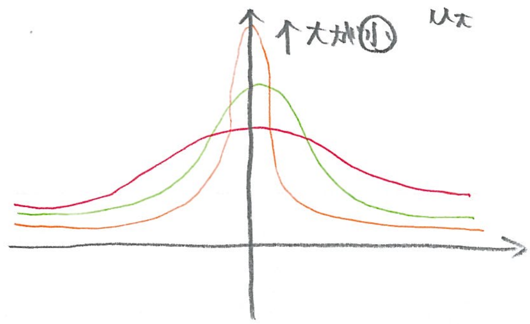
$$\therefore\quad \mathcal{F}(f*g)(\xi) = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(s)\, e^{-i(s+t)\xi}}_{} \, ds\, dt$$

$f(t)e^{-it\xi}$, $g(s)e^{-is\xi}$ に分ける.

$$= \hat{f}(\xi)\,\hat{g}(\xi)$$

定数 $t > 0$.

$$\mu_t(x) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}$$



$$\int |\mu_t(x)|\, dx = 1.$$

$$\mathcal{F}(\mu_t)(\xi) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}} \cdot e^{-ix\xi}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{y^2}{2}}\, e^{-i\sqrt{t}\, y\xi} \cdot \sqrt{t}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi}\, e^{-\frac{(\sqrt{t}\xi)^2}{2}} = e^{-\frac{t\xi^2}{2}}$$

$$(\mu_t * f)(x) = \int \mu_t(y) f(x-y)\, dy$$

$$= \int \frac{1}{2\pi} \int e^{-\frac{t\xi^2}{2}} e^{i\xi y}\, d\xi\, f(x-y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{t\xi^2}{2}} \left(\int_{-\infty}^{\infty} e^{i\xi y} f(x-y)\, dy \right) d\xi$$

$x - y = s$.

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{t\xi^2}{2}} \int_{\boxed{\infty}}^{\boxed{-\infty}} e^{i(x-s)\xi} f(s) (\boxed{-} ds)$$

$+ds$ にすると $\int_{-\infty}^{+\infty}$ になる

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{t\xi^2}{2}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-is\xi} f(s)\, ds\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t\xi^2}{2}}\,\hat{f}(\xi)\,e^{ix\xi}\,d\xi$$

ここで、$t \to 0$ とすると、

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\,e^{ix\xi}$$