Shannon

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \xi \in [-\pi, \pi) \\ 0 & \text{その他} \end{cases}$$

逆フーリエ

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi)\, e^{ix\xi} d\xi$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\xi} d\xi$$

$$= \frac{1}{2\pi}\left[\frac{1}{ix} e^{ix\xi}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi i x}\left\{e^{ix\pi} - e^{ix(-\pi)}\right\}$$

$$= \frac{\sin(\pi x)}{\pi x}$$

オイラーの公式

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\mathcal{F}\varphi(x-k) = \int_{-\infty}^{\infty} \varphi(x-k)\, e^{-ix\xi} dx$$

$$= \int_{-\infty}^{\infty} \varphi(y)\, e^{-iy\xi} \cdot e^{-ik\xi} dy \quad (\because y = x-k)$$

$$= e^{-ik\xi}\, \hat{\varphi}(\xi)$$

$\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ が正規直交系

$$\int_{-\infty}^{\infty} \varphi(x-k)\overline{\varphi(x-l)}\,dx = \delta_{k,l}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\varphi(x-k)}(\xi)\ \overline{\widehat{\varphi(x-l)}(\xi)}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}\hat{\varphi}(\xi)\ \overline{e^{-il\xi}\hat{\varphi}(\xi)}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\varphi}(\xi)|^2 \cdot e^{-i(k-l)\xi}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(k-l)\xi}\,d\xi$$

$k\neq l$ のとき

$$= \frac{-1}{2\pi i(k-l)}\left[e^{-i(k-l)\xi}\right]_{-\pi}^{\pi}$$

$$= \frac{-1}{2\pi i(k-l)}\left\{e^{-i(k-l)\pi} - e^{-i(k-l)(-\pi)}\right\}$$

$$= \frac{-2i\sin((k-l)\pi)}{2\pi i(k-l)} = \frac{-\sin((k-l)\pi)}{\pi(k-l)}$$

$$= 0 \quad (\because\ k-l\in\mathbb{Z} \text{ より } \sin((k-l)\pi)=0)$$

$k=l$ のとき

$$= \frac{1}{2\pi}\left[\pi-(-\pi)\right] = 1$$

$$\hat{f}(\xi) = \left( \sum_{k \in \mathbb{Z}} \langle \hat{f}, e_k \rangle_{L^2(-\pi,\pi)} \cdot e_k(\xi) \right) \hat{\varphi}(\xi)$$

$$\mathcal{F}^{-1} \downarrow \qquad \qquad \left\{ e_k(\xi) = \frac{1}{\sqrt{2\pi}} e^{-ik\xi} \right\}$$

$$f(x) = \sum_{k \in \mathbb{Z}} \langle \hat{f}, e_k \rangle_{L^2(-\pi,\pi)} \frac{1}{\sqrt{2\pi}} \operatorname{sinc}(x-k)$$

結論 → $= \sum f(k) \cdot \operatorname{sinc}(x-k)$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e_k(\xi)\, \hat{\varphi}(\xi)\, e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ik\xi} e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(x-k)\xi} d\xi$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{i(x-k)} \left[ e^{i(x-k)\pi} - e^{-i(x-k)\pi} \right]$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i(x-k)} \cdot 2i \sin((x-k)\pi)$$

$$= \operatorname{sinc}(x-k) \cdot \frac{1}{\sqrt{2\pi}} \qquad \left( \because \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \right)$$

$$\frac{1}{\sqrt{2\pi}} \langle \hat{f}, e_k \rangle_{L^2(-\pi,\pi)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\xi) \overline{e_k(\xi)} d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(\xi) \frac{1}{\sqrt{2\pi}} e^{ik\xi} d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ik\xi} d\xi$$

$$= f(k)$$

$f \in L^2(\mathbb{R})$

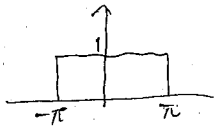
$\operatorname{supp} \hat{f}(\xi) \subset [-\pi, \pi]$

$(\operatorname{supp} f = \{x ; f(x) \neq 0\})$

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \operatorname{sinc}(x-k)$$

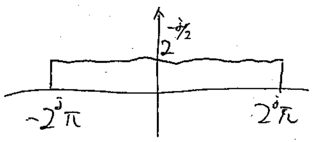
$$V_0 = \{ f \in L^2(\mathbb{R}) \ ; \ \operatorname{supp} \hat{f} \subset [-\pi, \pi) \}$$

の C.O.N.S が $\{ \varphi(x-k) \}_{k \in \mathbb{Z}}$




$$V_j = \{ f \in L^2(\mathbb{R}) \ ; \ \operatorname{supp} \hat{f} \subset [-2^j \pi, 2^j \pi) \}$$

の C.O.N.S が $\left\{ 2^{-j/2} \hat{\varphi}\left(\frac{\xi}{2^j}\right) e^{-\frac{ik\xi}{2^j}} \right\}_{k \in \mathbb{Z}}$




$V_j \subset V_{j+1}$

逆フーリエ

$$\frac{1}{2\pi} \int 2^{-j/2} \hat{\varphi}\left(\frac{\xi}{2^j}\right) e^{-\frac{ik\xi}{2^j}} e^{i\xi x} \, d\xi$$

$\frac{\xi}{2^j} = \eta$ とおく

$\frac{d\eta}{d\xi} = \frac{1}{2^j}$

$d\xi = 2^j d\eta$

$$= \frac{1}{2\pi} \cdot 2^{-j/2} \int \hat{\varphi}(\eta) e^{-ik\eta} \cdot e^{i 2^j \eta x} \cdot 2^j \, d\eta$$

$$= \frac{1}{2\pi} 2^{j/2} \int \hat{\varphi}(\eta) e^{i(2^j x - k)\eta} \, d\eta$$

$$= 2^{j/2} \varphi(2^j x - k)$$

$\{V_j\}_{j \in \mathbb{Z}}$ MRA

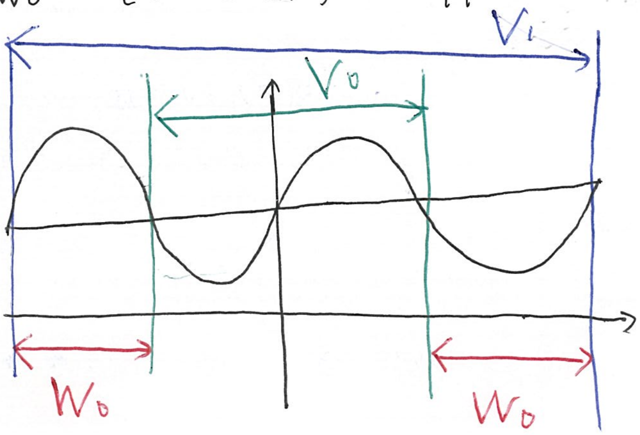
1. $V_j \subset V_{j+1}$
2. $f(x) \in V_0 \iff f(x-k) \in V_0 \quad k \in \mathbb{Z}$
3. $f(x) \in V_j \iff f(2x) \in V_{j+1}$
4. $\overline{\cup V_j} = L^2(\mathbb{R})$
5. $\cap V_j = \{0\}$

$\boxed{\text{MRA と } \varphi}$ からウェーブレットを作る。

$V_1 = V_0 \oplus W_0$

$\hat{W}_0$ の C.O.N.S が $\{\hat{\varphi}(\xi)e^{-ik\xi} \mid k\in\mathbb{Z}\}$

$W_0 = \{f\in L^2(\mathbb{R}) : \mathrm{supp}\,\hat{f} \subset [-2\pi,-\pi]\cup[\pi,2\pi]\}$




$W_0$ の C.O.N.S として $\{\psi(x-k)\}$ とするとき
$V_{j+1} = V_j \oplus W_j$ となる。

$W_j$ の C.O.N.S は $\{2^{j/2}\psi(2^jx-k) \mid k\in\mathbb{Z}\}$ である。

よって
$L^2(\mathbb{R})$ の C.O.N.S は $\{2^{j/2}\psi(2^jx-k) \mid j,k\in\mathbb{Z}\}$ になる。
$\psi$ : 正規直交ウェーブレット関数。

$$
\begin{aligned}
\psi(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(x)\cdot e^{ix\xi}\,d\xi \\
&= \frac{1}{2\pi}\left\{\int_{-2\pi}^{-\pi} e^{ix\xi}\,d\xi + \int_{\pi}^{2\pi} e^{ix\xi}\,d\xi\right\} \\
&= \frac{1}{2\pi}\left\{\frac{1}{ix}\left\{\left(e^{ix(-\pi)} - e^{ix(-2\pi)}\right) + \left(e^{ix2\pi} - e^{ix\pi}\right)\right\}\right\} \\
&= \frac{1}{2\pi ix}\left\{\left(e^{ix(-\pi)} - e^{ix\pi}\right) - \left(e^{ix(-2\pi)} - e^{ix2\pi}\right)\right\} \\
&= \frac{1}{2\pi ix}\left\{2i\sin(x\pi) - 2i\sin(2x\pi)\right\} \\
&= \frac{1}{\pi x}\left\{\sin(x\pi) - \sin(2x\pi)\right\}
\end{aligned}
$$

$\{g(x-k)\}$ が O.N.S である　　Orthonormal System　O.N.S

Complete Orthonormal System　C.O.N.S

$$\Updownarrow$$

$$\sum_{n\in\mathbb{Z}} |\hat{g}(\xi+2n\pi)|^2 = 1$$

$$\delta_{k,l} = \int g(x-k)\overline{g(x-l)}\,dx$$

$$= \frac{1}{2\pi}\int \hat{g}(\xi)e^{-ik\xi}\,\overline{\hat{g}(\xi)e^{-il\xi}}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 e^{-i(k-l)\xi}\,d\xi$$

$$= \frac{1}{2\pi}\sum_n \int_{2n\pi}^{2n\pi+2\pi} |\hat{g}(\xi)|^2 e^{-i(k-l)\xi}\,d\xi \qquad y = \xi - 2n\pi$$

$$= \frac{1}{2\pi}\sum_n \int_0^{2\pi} |\hat{g}(y+2n\pi)|^2 e^{-i(k-l)y+\cancel{2n\pi}}\,dy$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\boxed{\sum |\hat{g}(y+2n\pi)|^2}}_{G(y)}\,\underline{e^{-i(k-l)y}}\,dy$$

$$= \delta_{k,l}$$

$\varphi(x) \in V_0 \subset V_1$　$V_1$の CONS: $\{2^{1/2}\varphi(2x-k) \mid k\in\mathbb{Z}\}$

$\psi(x) \in W_0 \subset V_1$

$$\begin{cases} \varphi(x) = \sum h_k 2^{1/2}\varphi(2x-k) \\ h_k = \langle \varphi, 2^{1/2}\varphi(2x-k)\rangle = \int \varphi(x)\overline{2^{1/2}\varphi(2x-k)}\,dx \end{cases}$$

$$\hat{\varphi}(\xi) = \sum_k 2^{-1/2}h_k e^{-ik\frac{\xi}{2}}\hat{\varphi}(\tfrac{\xi}{2}) = m_0(\tfrac{\xi}{2})\hat{\varphi}(\tfrac{\xi}{2})$$

$$\left(m_0(\tfrac{\xi}{2}) = \sum_k 2^{-1/2}h_k e^{-ik\frac{\xi}{2}}\right)$$

$$\begin{cases} \psi(x) = \sum g_k 2^{1/2}\varphi(2x-k) \\ g_k = \langle \psi, 2^{1/2}\varphi(2x-k)\rangle = \int \psi(x)\overline{2^{1/2}\varphi(2x-k)}\,dx \end{cases}$$

$$\hat{\psi}(\xi) = \sum_k 2^{-1/2}g_k e^{-ik\frac{\xi}{2}}\hat{\varphi}(\tfrac{\xi}{2}) = m_1(\tfrac{\xi}{2})\hat{\varphi}(\tfrac{\xi}{2})$$

$$\left(m_1(\tfrac{\xi}{2}) = \sum_k 2^{-1/2}g_k e^{-ik\frac{\xi}{2}}\right)$$

$$\sum_{n\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2n\pi)\right|^2 = 1$$

$$= \sum_{n\in\mathbb{Z}} \left|m_0\left(\frac{\xi}{2} + n\pi\right)\hat{\varphi}\left(\frac{\xi}{2} + n\pi\right)\right|^2$$

$$= \sum_{n=2m} \left|m_0\left(\frac{\xi}{2} + 2n\pi\right)\hat{\varphi}\left(\frac{\xi}{2} + 2n\pi\right)\right|^2 + \sum_{n=2m+1}\left|m_0\left(\frac{\xi}{2} + 2n\pi + \pi\right)\hat{\varphi}\left(\frac{\xi}{2} + 2n\pi + \pi\right)\right|^2$$

$$= \left|m_0\left(\frac{\xi}{2}\right)\right|^2 \sum_{n=2m}\left|\hat{\varphi}\left(\frac{\xi}{2} + 2n\pi\right)\right|^2 + \left|m_0\left(\frac{\xi}{2} + \pi\right)\right|^2 \sum_{n=2M+1}\left|\hat{\varphi}\left(\frac{\xi}{2} + 2m\pi + \pi\right)\right|^2$$

$$= \left|m_0\left(\frac{\xi}{2}\right)\right|^2 \cdot 1 + \left|m_0\left(\frac{\xi}{2} + \pi\right)\right|^2 \cdot 1 = 1$$

$$\int \varphi(x-l)\,\overline{\psi(x-k)}\,dx = 0$$

$$\xrightarrow{\xi}\ \int \hat{\varphi}(\xi)e^{-ik\xi}\,\overline{\hat{\psi}(\xi)e^{-il\xi}}\,d\xi = 0$$

$$= \int m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right)e^{-ik\xi}\,\overline{m_1\left(\frac{\xi}{2}\right)}\,\overline{\hat{\varphi}\left(\frac{\xi}{2}\right)}\,e^{il\xi}\,d\xi$$

$$= \int_{-\infty}^{\infty} m_0\left(\frac{\xi}{2}\right)\overline{m_1\left(\frac{\xi}{2}\right)} \cdot \left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 \cdot e^{-i(k-l)\xi}\,d\xi$$

$$= \sum_{m\in\mathbb{Z}} \int_{2m\pi}^{2m\pi+2\pi} m_0\left(\frac{\xi}{2}\right)\overline{m_1\left(\frac{\xi}{2}\right)}\left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 e^{-i(k-l)\xi}\,d\xi$$

$\eta = \xi - 2n\pi$ とする。

$d\eta = d\xi$

$$= \sum_{m\in\mathbb{Z}} \int_0^{2\pi} m_0(\eta/2 + m\pi)\,\overline{m_1(\eta/2 + m\pi)}\,\left|\hat{\varphi}(\eta/2 + m\pi)\right|^2 e^{-i(k-l)(\eta + 2n\pi)}\,d\eta$$

$$= \int_0^{2\pi} \underset{\overset{\shortparallel}{0}}{\boxed{\sum_{m\in\mathbb{Z}} m_0(\eta/2 + m\pi)\,\overline{m_1(\eta/2 + m\pi)}\,\left|\hat{\varphi}(\eta/2 + m\pi)\right|^2}}\, e^{-i(k-l)\eta}\,d\eta = 0$$

$$\sum_{m\in\mathbb{Z}} m_0\left(\eta/2+m\pi\right)\overline{m_1\left(\eta/2+m\pi\right)}\left|\hat{\varphi}\left(\eta/2+m\pi\right)\right|^2$$

$$=\sum_{m=2n} m_0\left(\eta/2+2n\pi\right)\overline{m_1\left(\eta/2+2n\pi\right)}\left|\hat{\varphi}\left(\eta/2+2n\pi\right)\right|^2$$

$$+\sum_{n=m+1} m_0\left(\eta/2+2n\pi+\pi\right)\overline{m_1\left(\eta/2+2n\pi+\pi\right)}\left|\hat{\varphi}\left(\eta/2+2n\pi+\pi\right)\right|^2$$

$$= m_0\left(\eta/2\right)\overline{m_1\left(\eta/2\right)}\underbrace{\sum_{m=2n}\left|\hat{\varphi}\left(\eta/2+2n\pi\right)\right|^2}_{1}$$

$$+ m_0\left(\eta/2+\pi\right)\overline{m_1\left(\eta/2+\pi\right)}\underbrace{\sum_{m=2n+1}\left|\hat{\varphi}\left(\eta/2+2n\pi+\pi\right)\right|^2}_{1}=0$$

$\eta/2=\xi$ とする。

$$\therefore\ m_0(\xi)\overline{m_1(\xi)}+m_0(\xi+\pi)\overline{m_1(\xi+\pi)}=0$$

## 結論

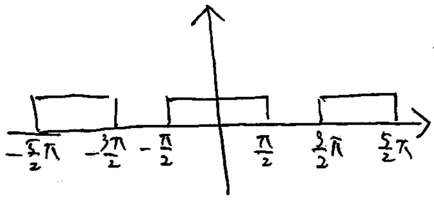
$$\begin{pmatrix} m_0(\xi) & m_1(\xi+\pi) \\ m_0(\xi) & m_1(\xi+\pi) \end{pmatrix}$$ がユニタリ行列

$$\times\begin{pmatrix} \overline{m_0(\xi)} & \overline{m_1(\xi)} \\ \overline{m_0(\xi+\pi)} & \overline{m_1(\xi+\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ が成り立つ

例

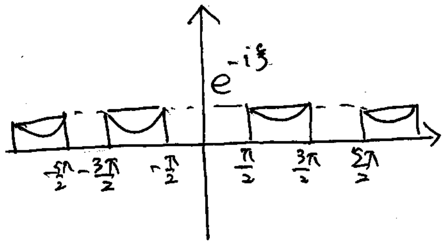
$$\boxed{m_1(\xi)=e^{i\xi}\,\overline{m_0(\xi+\pi)}}$$

が成り立って




$$\hat{\varphi}(\xi) = m_0\left(\frac{3}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right)$$

$$m_0(\xi) = \hat{\varphi}(2\xi)$$




$$m_1(\xi) = e^{-i\xi}\,\overline{m_0(\xi+\pi)}$$

$$\begin{aligned}
\hat{\psi}(\xi) &= m_1\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{3}{2}\right) \\
&= e^{-i\frac{\xi}{2}}\,\overline{m_0(\xi/2+\pi)}\,\hat{\varphi}\left(\frac{\xi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\psi(x) &= \frac{1}{2\pi}\int \hat{\psi}(\xi)\,e^{ix\xi}\,d\xi \\
&= \frac{1}{2\pi}\left\{\int_{-2\pi}^{-\pi} e^{(x-\frac{1}{2})i\xi}\,d\xi + \int_{\pi}^{2\pi} e^{(x-\frac{1}{2})i\xi}\,d\xi\right\} \\
&= \frac{1}{2\pi i(x-\frac{1}{2})}\left\{\left[e^{(x-\frac{1}{2})i\xi}\right]_{-2\pi}^{-\pi} + \left[e^{(x-\frac{1}{2})i\xi}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{1}{2\pi i(x-\frac{1}{2})}\left\{\left[e^{(x-\frac{1}{2})i\xi}\right]_{-2\pi}^{-\pi} + \left[e^{(x-\frac{1}{2})i\xi}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{1}{2\pi i(x-\frac{1}{2})}\left\{e^{-(x-\frac{1}{2})i\pi} - e^{-2(x-\frac{1}{2})i\pi} + e^{2(x-\frac{1}{2})i\pi} - e^{(x-\frac{1}{2})i\pi}\right\} \\
&= \frac{1}{\pi(x-\frac{1}{2})}\left\{\sin((2x-1)\pi) - \sin((x-\tfrac{1}{2})\pi)\right\} \\
&= 2\,\mathrm{sinc}(2x-1) - \mathrm{sinc}\left(x-\frac{1}{2}\right)
\end{aligned}$$