ROHLIN ACTIONS OF FINITE GROUPS ON THE RAZAK-JACELON ALGEBRA

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ABSTRACT. Let A be a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, and let α be a strongly outer action of a finite group G on A. In this paper, we show that $\alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{W}$ has the Rohlin property, where $\mathcal{W}$ is the Razak-Jacelon algebra. Combing this result with the recent classification results and our previous result, we see that such actions are unique up to conjugacy.

1. INTRODUCTION

Let $\mathcal{O}_2$ be the Cuntz algebra generated by 2 isometries. It is known that $\mathcal{O}_2$ is a simple separable unital nuclear purely infinite C*-algebra, and is KK-equivalent to $\{0\}$. Kirchberg and Phillips showed that a simple separable unital nuclear C*-algebra B is isomorphic to $\mathcal{O}_2$ if and only if B has an asymptotically central inclusion of $\mathcal{O}_2$ in [25]. In particular, if A is a simple separable unital nuclear C*-algebra, then $A \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$. It is known that $\mathcal{O}_2$ plays an important role in the classification of nuclear C*-algebra (see, for example, [15] and [41]).

Let $\mathcal{W}$ be the Razak-Jacelon algebra studied in [22], which is a certain simple separable nuclear stably projectionless C*-algebra having trivial K-groups and a unique tracial state and no unbounded traces. Note that $\mathcal{W}$ is KK-equivalent to $\{0\}$ and $\mathcal{O}_2$. Hence we may regard $\mathcal{W}$ as a stably finite analogue of $\mathcal{O}_2$. Combing Elliott, Gong, Lin and Niu's result [10] and Castillejos and Evington's result [2] (see also [3]), we see that if A is a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, then $A \otimes \mathcal{W}$ is isomorphic to $\mathcal{W}$. We refer the reader to [9], [10] (see also [12] and [18]) and [19] for recent progress in the classification of stably projectionless C*-algebras.

In the theory of operator algebras, the classification of group actions is one of the most fundamental problems and has a long history. There exists a complete classification of actions of countable amenable groups on approximately finite dimensional (AFD) factors. Although there exist some successes in the classification of group actions on "classifiable" C*-algebras, the classification of countable amenable group (outer) actions on "classifiable" C*-algebras is far from complete because of K-theoretical obstructions. We refer the reader to [21] and the references given there for details and results in the classification of group actions on operator algebras. We shall review only some results that are directly related to this paper.

Connes [5] classified finite cyclic group actions on the AFD factor $\mathcal{R}_0$ of type II$_1$ up to conjugacy. More generally, Jones [23] classified finite group actions on $\mathcal{R}_0$. In particular, outer actions of a finite group on $\mathcal{R}_0$ are unique up to conjugacy.

In [20], Izumi introduced the Rohlin property of finite group actions on unital C*-algebras and showed an equivariant version of the Kirchberg-Phillips type theorem

for finite group actions on $\mathcal{O}_2$. Indeed, he characterized Rohlin actions on $\mathcal{O}_2$ by using the fixed point subalgebra of the central sequence C*-algebra of $\mathcal{O}_2$ and showed that if α is an outer action of a finite group G on a simple separable unital nuclear C*-algebra A, then $\alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{O}_2$ has the Rohlin property. In particular, such actions are unique up to conjugacy. Note that Izumi also showed that there exist uncountably many mutually non-conjugate outer actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$. Also, Goldstein and Izumi obtained an equivariant Kirchberg-Phillips type result for finite group actions on $\mathcal{O}_\infty$ in [17].

Remarkably, Szabó generalized Izumi's result to countable amenable group actions in [49]. He showed that countable amenable group outer actions on $\mathcal{O}_2$ that equivariantly absorb the trivial action on $\mathcal{O}_2$ are unique up to strong cocycle conjugacy. Note that Szabó considered more general settings and obtained results for strongly self-absorbing C*-dynamical systems. See [46], [47], [48], [49], [50] and [51].

In this paper, we shall consider an equivariant Kirchberg-Phillips type result for finite group actions on $\mathcal{W}$. Indeed, we shall show that if α is a strongly outer action of a finite group G on a simple separable nuclear C*-algebra A with a unique tracial state and no unbounded traces, then $\alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{W}$ has the Rohlin property (Theorem 6.4). Since the author showed that Rohlin actions of a finite group on $\mathcal{W}$ are unique up to conjugacy in [36], we see that such actions are unique up to conjugacy by Elliott, Gong, Lin and Niu's result and Castillejos and Evington's result. Indeed, we obtain the following theorem.

Theorem. (Corollary 6.5)
Let A and B be simple separable nuclear C*-algebras with a unique tracial state and no unbounded traces, and let α and β be strongly outer actions of a finite group G on A and B, respectively. Then $\alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{W}$ is conjugate to $\beta \otimes \mathrm{id}$ on $B \otimes \mathcal{W}$.

Our main result (Theorem 6.4) is shown by using a cohomolgy vanishing type result (Lemma 6.2). The proof of Lemma 6.2 is based on Connes' 2×2 matrix trick in [5, Corollary 2.6]. We need to consider the comparison theory for projections in the fixed point subalgebra $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ of the central sequence C*-algebra of $A \otimes \mathcal{W}$ for Connes' 2×2 matrix trick. We obtain this as a corollary of a classification up to unitary equivalence of certain normal elements in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$. This classification is based on arguments in [37] where the author classified certain unitary elements and projections in $F(\mathcal{W})$ up to unitary equivalence.

This paper is organized as follows. In Section 2, we collect notations, definitions and some results. In Section 3 and Section 4, we show a variant of [37, Corollary 3.8], which is a main technical tool in this paper. In particular, we introduce a (non-separable) C*-algebra $\mathcal{B}^\gamma$, and show that $\mathcal{B}^\gamma$ has strict comparison (Proposition 3.8) in Section 3. Note that $\mathcal{B}^\gamma$ is a target algebra of a (natural) homomorphism from $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}} \otimes \mathcal{W}$. The proof of Proposition 3.8 is essentially based on arguments in [32], [33] and [34]. In particular, it is important to consider the property (SI) and the weak Rohlin property. These concepts were introduced by Sato in his pioneering work [43] and [44] (see also [27]). We refer the reader to [45] for recent progress of such type arguments. Section 4 is essentially based on arguments in [12] (see [37, Section 3]). In Section 5, we classify certain normal elements in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ up to unitary equivalence (Theorem 5.3), and show a comparison theorem for certain projections in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ (Corollary 5.5). In Section 6, we show the main result in this paper.

2. Preliminaries

In this section we shall collect notations, definitions and some results.

For a C*-algebra A, let A_+ denote the set of positive elements in A and $A_{+,1}$ the set of positive contractions in A. For $x, y \in A$, let $[x, y]$ be the commutator $xy - yx$. We denote by $K(H)$ and M_{n^∞} for $n \in \mathbb{N}$ the C*-algebra of compact operators on a Hilbert space H and the uniformly hyperfinite (UHF) algebra of type n^∞, respectively.

2.1. **Approximate units and actions.** If A is a separable C*-algebra, then there exists a positive element $s \in A$ such that sA is dense in A. Such a positive element s is said to be *strictly positive* in A. For any $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(t) := \begin{cases} 0 & t \in [0, \frac{1}{n+1}] \\ n(n+1)t - n & t \in (\frac{1}{n+1}, \frac{1}{n}] \\ 1 & t \in (\frac{1}{n}, 1] \end{cases} .$$

If s is a strictly positive element in A and $\|s\| = 1$, then $\{f_n(s)\}_{n \in \mathbb{N}}$ is an approximate unit for A with $f_{n+1}(s)f_n(s) = f_n(s)$. Let $A^\sim$ denote the unitization algebra of A. Note that we assume $A^\sim = A$ if A is unital. Let $M(A)$ be the *multiplier algebra* of A, which is the largest unital C*-algebra that contains A as an essential ideal. If α is an automorphism of A, then α extends uniquely to an automorphism of $M(A)$. We denote it by the same symbol α for simplicity.

We denote by $\mathrm{Aut}(A)$ the automorphism group of A. An automorphism α of A is said to be *inner* if there exists a unitary element u in $M(A)$ such that $\alpha(x) = \mathrm{Ad}(u)(x) = uxu^*$ for any $x \in A$. For a subset F of A and $\varepsilon > 0$, we say a completely positive (c.p.) map $\varphi : A \to B$ is (F, ε)-*multiplicative* if

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| < \varepsilon$$

for any $x, y \in F$.

An *action* α of a discrete group G on A is a homomorphism from G to $\mathrm{Aut}(A)$. We say that α is *outer* if α_g is not inner for any $g \in G \setminus \{\iota\}$ where ι is the identity of G. An α-*cocycle* is a map w from G to the unitary group of $M(A)$ such that $w(gh) = w(g)\alpha_g(w(h))$ for any $g, h \in G$. We say that an α-cocycle w is a *coboundary* if there exists a unitary element v in $M(A)$ such that $w(g) = v\alpha_g(v^*)$ for any $g \in G$. For two G-actions α on A and β on B, we say that α and β are *conjugate* if there exists an isomorphism θ from A onto B such that $\theta \circ \alpha_g = \beta_g \circ \theta$ for any $g \in G$.

Every tracial state τ on A extends uniquely to a tracial state on $M(A)$. We denote it by the same symbol τ for simplicity. Let (π_τ, H_τ) be the Gelfand-Naimark-Segal (GNS) representation of A associated with τ. Then τ extends uniquely to a normal tracial state $\tilde{\tau}$ on $\pi_\tau(A)''$. If α is an automorphism of A such that $\tau \circ \alpha = \tau$, then α extends uniquely to an automorphism $\tilde{\alpha}$ of $\pi_\tau(A)''$. Moreover if α is an action of G on A such that $\tau \circ \alpha_g = \tau$ for any $g \in G$, then α extends uniquely to a von Neumann algebraic action $\tilde{\alpha}$ on $\pi_\tau(A)''$. We say that an action α of G on a C*-algebra A with a unique tracial state τ is *strongly outer* if $\tilde{\alpha}_g$ is not inner in $\pi_\tau(A)''$ for any $g \in G \setminus \{\iota\}$.

2.2. **Kirchberg's central sequence C*-algebras.** We shall recall Kirchberg's central sequence C*-algebras in [24] (see also [35, Section 5] and [37, Section 2.2]). Fix a free ultrafilter ω on $\mathbb{N}$. For a C*-algebra A, put

$$c_\omega(A) := \{\{x_n\}_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n \to \omega} \|x_n\| = 0\}, \ A^\omega := \ell^\infty(\mathbb{N}, A)/c_\omega(A).$$

A sequence $(x_n)_n$ is a representative of an element in A^ω. Let B be a C*-subalgebra of A. We identify A and B with the C*-subalgebras of A^ω consisting of equivalence classes of constant sequences. Set

$$A_\omega := A^\omega \cap A', \ \mathrm{Ann}(B, A^\omega) := \{(x_n)_n \in A^\omega \cap B' \mid (x_n)_n b = 0 \text{ for any } b \in B\}.$$

Then $\mathrm{Ann}(B, A^\omega)$ is a closed ideal of $A^\omega \cap B'$. Define a *central sequence C*-algebra* $F(A)$ of A by

$$F(A) := A_\omega / \mathrm{Ann}(A, A^\omega).$$

If $\{h_n\}_{n \in \mathbb{N}}$ is a countable approximate unit for A, then $[(h_n)_n]$ is a unit in $F(A)$. It can be easily checked that $F(A)$ is isomorphic to $M(A)^\omega \cap A' / \mathrm{Ann}(A, M(A)^\omega)$ and $A^\sim_\omega / \mathrm{Ann}(A, (A^\sim)^\omega)$. If α is an automorphism of A, α induces natural automorphisms of A^ω, A_ω and $F(A)$. We denote them by the same symbol α for simplicity. For a tracial state τ on A, define $\tau_\omega([(x_n)_n]) := \lim_{n \to \omega} \tau(x_n)$. Then τ_ω is a well defined tracial state on $F(A)$ by [37, Proposition 2.1].

2.3. **Razak-Jacelon algebra.** Let $\mathcal{W}$ be the Razak-Jacelon algebra studied in [22], which is a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, and is KK-equivalent to $\{0\}$. The Razak-Jacelon algebra $\mathcal{W}$ is constructed as an inductive limit C*-algebra of Razak's building block in [39]. Let S_1 and S_2 be the generators of the Cuntz algebra $\mathcal{O}_2$. For every $\lambda_1, \lambda_2 \in \mathbb{R}$, define a flow γ on $\mathcal{O}_2$ by $\gamma_t(S_j) = e^{it\lambda_j} S_j$. Kishimoto and Kumjian showed that if λ_1 and λ_2 are all non-zero, of the same sign and λ_1 and λ_2 generate $\mathbb{R}$ as a closed subgroup, then $\mathcal{O}_2 \rtimes_\gamma \mathbb{R}$ is a simple stably projectionless C*-algebra with unique (up to scalar multiple) trace in [28] and [29]. Robert showed that $\mathcal{W} \otimes K(\ell^2(\mathbb{N}))$ is isomorphic to $\mathcal{O}_2 \rtimes_\gamma \mathbb{R}$ for some λ_1 and λ_2 in [40]. (See also [8].) Razak's classification theorem [39] implies that $\mathcal{W}$ is UHF-stable, and hence $\mathcal{W}$ is $\mathcal{Z}$-stable.

2.4. **Corollaries of Matui and Sato's results.** We shall collect some corollaries of Matui and Sato's results in [31] and [33]. Although they assume that C*-algebras are unital, their arguments for the following results work for non-unital C*-algebras by suitable modifications (see [35] and [37]).

First, we recall the definition of the weak Rohlin property. See [31, Definition 2.7] and [33, Definition 2.5]. Note that Matui and Sato define the weak Rohlin property for more general settings.

Definition 2.1. Let A be a simple C*-algebra with a unique tracial state τ, and let α be an action of a finite group G on A. We say that α has the weak Rohlin property if there exists an positive contraction f in $F(A)$ such that

$$\alpha_g(f)\alpha_h(f) = 0, \quad \tau_\omega(f) = \frac{1}{|G|}$$

for any $g, h \in G$ with $g \neq h$.

Essentially the same proof as [31, Theorem 3.4] shows the following theorem. See also the proof of [37, Lemma 6.2] and [33, Theorem 3.6].

Theorem 2.2. Let A be a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, and let α be an action of a finite group G on A. Then α has the weak Rohlin property if and only if α is strongly outer.

Essentially the same proofs as [33, Lemma 4.7] and [33, Proposition 4.8] show the following proposition. See also [34, Propostion 3.3] and [1, Theorem 4.1]. Note that if A is a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, then $A \otimes \mathcal{W}$ has property (SI) since $\mathcal{W}$ is $\mathcal{Z}$-stable (see [42], [32] and [35]).

Proposition 2.3. Let A be a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, and let α be a strongly outer action of a finite group G on A. Then:
(i) $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ has a unique tracial state τ_ω.
(ii) If a and b are positive elements in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ satisfying $d_{\tau_\omega}(a) < d_{\tau_\omega}(b)$, then there exists an element $r \in F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ such that $r^* b r = a$.

2.5. **Rohlin property and properties of** $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$**.** We shall recall some results in [36] (see also [16]) and [37].

Definition 2.4. (cf. [20, Definition 3.1] and [36, Definition 3.1]). An action α of a finite group G on a separable C*-algebra A is said to have the *Rohlin property* if there exists a partition of unity $\{p_g\}_{g \in G} \subset F(A)$ consisting of projections satisfying

$$\alpha_g(p_h) = p_{gh},$$

for any $g, h \in G$.

For any finite group G, there exists an action of G on $\mathcal{W}$ with the Rohlin property by [36, Example 3.2]. The following theorem is [36, Corollary 3.7].

Theorem 2.5. Let α and β be actions of a finite group G on $\mathcal{W}$ with the Rohlin property. Then α and β are conjugate.

Note that there exists a strongly outer action α of $\mathbb{Z}_2$ on $\mathcal{W}$ such that α does not have the Rohlin property (see [36, Example 5.6]).

Since we can regard $F(\mathcal{W})$ is a unital C*-subalgebra of $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$, we obtain the following proposition by [37, Proposition 4.2] and Proposition 2.3.

Proposition 2.6. Let τ_ω be the unique tracial state on $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$.
(i) For any $N \in \mathbb{N}$, there exists a unital homomorphism from $M_N(\mathbb{C})$ to $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$.
(ii) For any $\theta \in [0, 1]$, there exists a non-zero projection p in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ such that $\tau_\omega(p) = \theta$.
(iii) Let h be a positive element in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ such that $d_{\tau_\omega}(h) > 0$. For any $\theta \in [0, d_{\tau_\omega}(h))$, there exists a non-zero projection p in $\overline{h F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}} h}$ such that $\tau_\omega(p) = \theta$.

Using the proposition above instead of [37, Proposition 4.2], the same arguments as in [37, Section 4] show the following proposition.

Proposition 2.7. (cf. [37, Proposition 4.8]). Let p and q be projections in $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$ such that $\tau_\omega(p) < 1$ where τ_ω is the unique tracial state on $F(A \otimes \mathcal{W})^{\alpha \otimes \mathrm{id}}$. Then p and q are Murray-von Neumann equivalent if and only if p and q are unitarily equivalent.

3. Target algebra

In the rest of this paper, we assume that A is a simple separable nuclear C*-algebra with a unique tracial state τ_A and no unbounded traces, and α is a strongly outer action of a finite group G on A. Define an action γ on $A \otimes \mathcal{W}$ by $\gamma := \alpha \otimes \mathrm{id}$. Let $\tau_{\mathcal{W}}$ denote the unique tracial state on $\mathcal{W}$, and let $\tau := \tau_A \otimes \tau_{\mathcal{W}}$ on $A \otimes \mathcal{W}$. For any $a \in A$ and $b \in \mathcal{W}$, we regard $a \otimes 1_{\mathcal{W}^\sim}$ and $1_{A^\sim} \otimes b$ as elements in $M(A \otimes \mathcal{W})$. Put

$$\mathcal{A} := \{(x_n)_n \in (A \otimes \mathcal{W})^\omega \mid ([x_n, a \otimes 1_{\mathcal{W}^\sim}])_n = 0 \text{ for any } a \in A\}$$

and

$$\mathcal{I} := \{(x_n)_n \in \mathcal{A} \mid (x_n(a \otimes 1_{\mathcal{W}^\sim}))_n = 0 \text{ for any } a \in A\}.$$

Then $\mathcal{I}$ is a closed ideal of $\mathcal{A}$, and define $\mathcal{B} := \mathcal{A}/\mathcal{I}$. Note that for any $[(x_n)_n] \in \mathcal{B}$,

$$\|[(x_n)_n]\| = \sup_{a \in A_{+,1}} \lim_{n \to \omega} \|x_n(a \otimes 1_{\mathcal{W}^\sim})\|.$$

Indeed, let $\|[(x_n)_n]\|' := \sup_{a \in A_{+,1}} \lim_{n \to \omega} \|x_n(a \otimes 1_{\mathcal{W}^\sim})\|$ for any $[(x_n)_n] \in \mathcal{B}$. Then it can be easily checked that $\| \cdot \|'$ is a well defined C*-norm on $\mathcal{B}$. By the uniqueness of the C*-norm, $\|[(x_n)_n]\| = \|[(x_n)_n]\|'$ for any $[(x_n)_n] \in \mathcal{B}$. The action γ on $A \otimes \mathcal{W}$ induces a natural action on $\mathcal{B}$. We denote it by the same symbol γ

for simplicity. In this section we shall consider properties of the fixed point algebra $\mathcal{B}^\gamma$.

Consider the GNS representation (π_τ, H_τ) of $A \otimes \mathcal{W}$ associated with τ. Note that π_τ extends to a representation $\overline{\pi}_\tau$ of $M(A \otimes \mathcal{W})$ on H_τ and $\overline{\pi}_\tau(M(A \otimes \mathcal{W})) \subset \pi_\tau(A \otimes \mathcal{W})''$ (see, for example, [38, 3.12]). Put

$$M := \ell^\infty(\mathbb{N}, \pi_\tau(A \otimes \mathcal{W})'')/\{\{x_n\}_{n \in \mathbb{N}} \mid \lim_{n \to \omega} \tilde{\tau}(x_n^* x_n) = 0\},$$

and define a homomorphism Π from A to M by $\Pi(a) := (\overline{\pi}_\tau(a \otimes 1_{\mathcal{W}^\sim}))_n$. Note that M is a von Neumann algebraic ultrapower of $\pi_\tau(A \otimes \mathcal{W})''$. Since $\tau = \tau_A \otimes \tau_\mathcal{W}$, $\pi_\tau(A \otimes \mathcal{W})''$ is isomorphic to $\pi_{\tau_A}(A)'' \overline{\otimes} \pi_{\tau_\mathcal{W}}(\mathcal{W})''$. Moreover, $\pi_\tau(A \otimes \mathcal{W})''$, $\pi_{\tau_A}(A)''$ and $\pi_{\tau_\mathcal{W}}(\mathcal{W})''$ are isomorphic to the AFD II_1 factor $\mathcal{R}_0$. Set

$$\mathcal{M} := M \cap \Pi(A)'.$$

It is easy to see that $\mathcal{M}$ is isomorphic to $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0)^\omega \cap (\mathcal{R}_0 \overline{\otimes} \mathbb{C})'$ where $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0)^\omega$ is the von Neumann algebraic ultrapower of $\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0$.

Proposition 3.1. With notation as above, $\mathcal{M}$ is a factor of type II_1.

Proof. Let $\{N_n\}_{n=1}^\infty$ be an increasing sequence of finite-dimensional subfactors such that $\mathcal{R}_0 = (\bigcup_{n=1}^\infty N_n)''$. Since $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0) \cap (N_n \overline{\otimes} \mathbb{C})' = (\mathcal{R}_0 \cap N_n') \overline{\otimes} \mathcal{R}_0$ is a factor of type II_1, the same proof as in [52, Theorem XIV.4.18] shows this proposition. $\square$

The action $\tilde{\gamma} = \tilde{\alpha} \otimes \mathrm{id}$ on $\pi_\tau(A \otimes \mathcal{W})'' \cong \pi_{\tau_A}(A)'' \overline{\otimes} \pi_{\tau_\mathcal{W}}(\mathcal{W})''$ induces an action on $\mathcal{M}$. We denote it by the same symbol $\tilde{\gamma}$ for simplicity. The following lemma is essentially based on [4, Proposition 2.1.2].

Lemma 3.2. The action $\tilde{\gamma}$ on $\mathcal{M}$ is outer.

Proof. It is enough to show that for any element $(u_n)_n$ in $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0)^\omega \cap (\mathcal{R}_0 \overline{\otimes} \mathbb{C})'$, there exists an element $(x_n)_n$ in $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0)^\omega \cap (\mathcal{R}_0 \overline{\otimes} \mathbb{C})'$ such that $(\tilde{\gamma}(x_n))_n \neq (x_n)_n$ and $[(x_n)_n, (u_n)_n] = 0$.

Let $(u_n)_n$ be an element in $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0)^\omega \cap (\mathcal{R}_0 \overline{\otimes} \mathbb{C})'$. By [52, Theorem XIV.4.16], there exists an element $(a_n)_n$ in $\mathcal{R}_0^\omega \cap \mathcal{R}'$ such that $(\tilde{\alpha}(a_n))_n \neq (a_n)_n$ because $\tilde{\alpha}$ is outer and $\mathcal{R}_0$ is the AFD II_1 factor. Put $(x_n)_n := (a_n \otimes 1_{\mathcal{R}_0})_n$ in $(\mathcal{R}_0 \overline{\otimes} \mathcal{R}_0)^\omega$. Then $(\tilde{\gamma}(x_n))_n \neq (x_n)_n$ and $[(x_n)_n, y] = 0$ for any $y \in \mathcal{R}_0 \overline{\otimes} \mathcal{R}_0$. Taking a suitable subsequence of $(x_n)_n$, we obtain the conclusion. $\square$

By Proposition 3.1 and Lemma 3.2, we obtain the following proposition.

Proposition 3.3. The fixed point algebra $\mathcal{M}^{\tilde{\gamma}}$ is a factor of type II_1.

Define a homomorphism Φ from $(A \otimes \mathcal{W})^\omega$ to M by $\Phi((x_n)_n) := (\pi_\tau(x_n))_n$. By Kaplansky's density theorem, we see that Φ is surjective. It is easy to see that Φ maps $\mathcal{A}$ into $\mathcal{M}$. The following proposition is essentially based on [26, Theorem 3.3] and [34, Theorem 3.1].

Proposition 3.4. The restriction $\Phi|_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}$ is surjective.

Proof. Let x be a contraction in $\mathcal{M}$. Then there exists a contraction $(x_n)_n$ in $(A \otimes \mathcal{W})^\omega$ such that $\Phi((x_n)_n) = x$. Let D be a C*-subalgebra of $(A \otimes \mathcal{W})^\omega$ generated by $\{((a \otimes 1_{\mathcal{W}^\sim})x_n)_n, (x_n(a \otimes 1_{\mathcal{W}^\sim}))_n \mid a \in A\}$, and put $I := \ker \Phi|_D$. Then the rest of proof is same as the proof of [34, Theorem 3.1]. $\square$

Let $\{h_n\}_{n \in \mathbb{N}}$ be an approximate unit for A. Since $\lim_{n \to \infty} \tau(h_n \otimes 1_\mathcal{W}) = 1$, a similar argument as in the proof of [37, Proposition 2.1] shows $\mathcal{I} \subset \ker \Phi|_{\mathcal{A}}$. Therefore $\Phi|_{\mathcal{A}}$ induces a surjective homomorphism ϱ from $\mathcal{B}$ to $\mathcal{M}$. Since γ is an action of a finite group, it is easy to show the following proposition.

Proposition 3.5. The restriction $\varrho|_{\mathcal{B}^\gamma} : \mathcal{B}^\gamma \to \mathcal{M}^{\tilde{\gamma}}$ is surjective.

The following lemma is essentially based on [34, Lemma 3.2]. This lemma may be considered that a homomorphism $a \mapsto a \otimes 1_{\mathcal{W}^\sim}$ from A to $M(A \otimes \mathcal{W})^\omega$ has "property (SI) with respect to $A \otimes \mathcal{W}$".

Lemma 3.6. Let $(x_n)_n$ and $(y_n)_n$ be positive contractions in $\mathcal{A}$ such that
$$\lim_{n \to \omega} \tau(x_n) = 0 \quad \text{and} \quad \inf_{m \in \mathbb{N}} \lim_{n \to \omega} \tau(y_n^m) > 0.$$
Then there exists an element $(s_n)_n$ in $\mathcal{A}$ such that $(s_n^* s_n)_n = (x_n)_n$ and $(y_n s_n)_n = (s_n)_n$.

Proof. Similar arguments as in the proofs of [34, Lemma 3.2] and [32, Theorem 1.1] with some modifications in [35, Section 5] show this lemma. Indeed, let φ be a pure state on A. We can uniquely extend φ to a pure state $\tilde{\varphi}$ on $A^\sim$. Since we may assume that A is a (separable simple) non-type I C*-algebra, $K(H_{\tilde{\varphi}}) \cap \pi_{\tilde{\varphi}}(A^\sim) = \{0\}$. Therefore [26, Proposition 5.9] implies that the identity map on $A^\sim$ can be approximated in the pointwise norm topology by a completely positive map ψ of the form
$$\psi(a) = \sum_{i,j=1} \tilde{\varphi}(d_i^* a d_j) c_i^* c_j, \quad a \in A^\sim,$$
where $c_i, d_i \in A^\sim$. Note that $\sum_{i,j=1} \tilde{\varphi}(d_i^* a d_j)(c_i^* c_j \otimes 1_{\mathcal{W}^\sim}) x_n$ is an element in $A \otimes \mathcal{W}$. Since $A \otimes \mathcal{W}$ has strict comparison, a similar argument as in the proof of [34, Lemma 3.2] (we need to use [35, Lemma 5.7]) shows that there exists a sequence of $(s_n)_n$ in $A \otimes \mathcal{W}$ such that $(f_n s_n)_n = (s_n)_n$ and $(s_n^*(a \otimes 1_{\mathcal{W}^\sim}) s_n)_n = ((a \otimes 1_{\mathcal{W}^\sim}) e_n)$ in $(A \otimes \mathcal{W})^\omega$ for any $a \in A^\sim$. Therefore we obtain the conclusion (see [35, Remark 5.5]). $\square$

For any $[(x_n)_n] \in \mathcal{B}$, let $\tau_{\mathcal{B}}([(x_n)_n]) := \lim_{n \to \omega} \tau(x_n)$. By a similar argument as in the proof of [37, Proposition 2.1], $\tau_{\mathcal{B}}$ is a well defined tracial state on $\mathcal{B}$. The following proposition is essentially based on [33, Proposition 4.5]. See also the proof of [31, Theorem 4.7].

Proposition 3.7. Let x and y be positive contractions in $\mathcal{B}^\gamma$ such that
$$\tau_{\mathcal{B}}(x) = 0 \quad \text{and} \quad \inf_{m \in \mathbb{N}} \tau_{\mathcal{B}}(y^m) > 0.$$
Then there exists an element s in $\mathcal{B}^\gamma$ such that $s^* s = x$ and $ys = s$.

Proof. Let $(x_n)_n$ and $(y_n)_n$ be positive contractions in $\mathcal{A}$ such that $x = [(x_n)_n]$ and $y = [(y_n)_n]$. Then we have
$$(\gamma_g(x_n) - x_n)_n(a \otimes 1_{\mathcal{W}^\sim}) = 0 \quad \text{and} \quad (\gamma_g(y_n) - y_n)_n(a \otimes 1_{\mathcal{W}^\sim}) = 0$$
for any $a \in A$ and $g \in G$. Since α is strongly outer, Theorem 2.2 implies that there exists a positive contraction $(f_n)_n$ in A_ω such that
$$(\alpha_g(f_n)\alpha_h(f_n))_n a = 0 \quad \text{and} \quad \lim_{n \to \omega} \tau_A(f_n) = \frac{1}{|G|}$$
for any $a \in A$ and $g, h \in G$ with $g \neq h$. Let $\{k_n\}_{n=1}^\infty$ be an approximate unit for $\mathcal{W}$. Then we have $(f_n \otimes k_n)_n \in \mathcal{A}$,
$$\lim_{n \to \omega} \gamma_g(f_n \otimes k_n)\gamma_h(f_n \otimes k_n)(a \otimes 1_{\mathcal{W}^\sim}) = 0 \quad \text{and} \quad \lim_{n \to \omega} \tau(f_n \otimes k_n) = \frac{1}{|G|}$$
for any $a \in A$ and $g, h \in G$ with $g \neq h$. Using [35, Lemma 5.6] instead of [32, Lemma 4.6], a similar argument as in the proof of [33, Proposition 4.5] shows that there exists a positive contraction $(\tilde{y}_n)_n$ in $\mathcal{A}$ such that
$$(\tilde{y}_n)_n \leq (y_n)_n, \quad \inf_{m \in \mathbb{N}} \lim_{n \to \omega} \tau(\tilde{y}_n^m) > 0 \quad \text{and} \quad \lim_{n \to \omega} \gamma_g(\tilde{y}_n)\gamma_h(\tilde{y}_n)(a \otimes 1_{\mathcal{W}^\sim}) = 0$$

for any $a \in A$ and $g, h \in G$ with $g \neq h$. By Lemma 3.6, there exists an element $(r_n)_n$ in $\mathcal{A}$ such that $(r_n^* r_n)_n = (x_n)_n$ and $(\tilde{y}_n r_n)_n = (r_n)_n$. Since $(y_n)_n$ is a positive contraction and $(\tilde{y}_n)_n \leq (y_n)_n$, we have $(y_n r_n)_n = (r_n)_n$. Put

$$(s_n)_n := \frac{1}{|G|} \sum_{g \in G} \gamma_g((r_n)_n) \in \mathcal{A}.$$

Then we have

$$(\gamma_g(s_n) - s_n)_n = 0, \quad (s_n^* s_n - x_n)_n (a \otimes 1_{\mathcal{W}\sim}) = 0 \quad \text{and} \quad (y_n s_n - s_n)_n (a \otimes 1_{\mathcal{W}\sim}) = 0$$

for any $a \in A$ and $g \in G$. Therefore, putting $s := [(s_n)_n] \in \mathcal{B}^\gamma$, we obtain the conclusion. $\qquad\square$

The following proposition is essentially based on [33, Proposition 4.8] and [34, Proposition 3.3].

Proposition 3.8. (i) $\tau_\mathcal{B}$ is the unique tracial state on $\mathcal{B}^\gamma$.
(ii) $\mathcal{B}^\gamma$ has strict comparison.

Proof. (i) By Proposition 3.3 and Proposition 3.5, it suffices to show that if $[(x_n)_n]$ is a positive contraction in $\ker \varrho|_{\mathcal{B}^\gamma}$, then $T([(x_n)_n]) = 0$ for any tracial state T on $\mathcal{B}^\gamma$. Note that $[(x_n)_n]^{1/2} \in \ker \varrho|_{\mathcal{B}^\gamma}$, and hence $\tau_\mathcal{B}([(x_n)_n]) = 0$. Let $\{e_n\}_{n=1}$ be an approximate unit for $A \otimes \mathcal{W}$. Then it is easy to see that for any $m \in \mathbb{N}$, $\tau_\mathcal{B}(([(e_n)_n] - [(x_n)_n])^m) = 1$. By Proposition 3.7, there exists an element $s_1 \in \mathcal{B}^\gamma$ such that $s_1^* s_1 = [(x_n)_n]$ and $([(e_n)_n] - [(x_n)_n]) s_1 = s_1$. Hence we have $s_1 s_1^* \leq [(e_n)_n] - [(x_n)_n]$. Since $[(x_n)_n] + s_1 s_1^*$ is a positive contraction and $\tau_\mathcal{B}([(x_n)_n] + s_1 s_1^*) = 0$, the same argument as above shows that there exists an element $s_2 \in \mathcal{B}^\gamma$ such that $s_2^* s_2 = [(x_n)_n]$ and $([(e_n)_n] - [(x_n)_n] - s_1 s_1^*) s_2 = s_2$. Repeating this process, for any $N \in \mathbb{N}$, we obtain elements $s_1, s_2, ..., s_N$ in $\mathcal{B}^\gamma$ such that

$$s_i^* s_i = [(x_n)_n] \quad \text{and} \quad [(x_n)_n] + \sum_{i=1}^N s_i s_i^* \leq [(e_n)_n].$$

Since T is a tracial state and $[(e_n)_n]$ is a contraction, $(N+1)T([(x_n)_n]) \leq 1$. Therefore $T([(x_n)_n]) = 0$.

(ii) Since $\mathcal{W} \otimes M_n(\mathbb{C})$ is isomorphic to $\mathcal{W}$, it can be easily checked that $\mathcal{B}^\gamma \otimes M_n(\mathbb{C})$ is isomorphic to $\mathcal{B}^\gamma$. Hence it is enough to show that if a and b are positive elements in $\mathcal{B}^\gamma$ with $d_{\tau_\mathcal{B}}(a) < d_{\tau_\mathcal{B}}(b)$, then there exists an element r in $\mathcal{B}^\gamma$ such that $r^* b r = a$. Using Proposition 3.3, Proposition 3.5 and Proposition 3.7 instead of [33, Lemma 4.2], [33, Theorem 4.3] and [33, Proposition 4.5], the same argument as in the proof of [33, Proposition 4.8] shows this. Therefore the proof is complete. $\qquad\square$

4. Stable uniqueness theorem

In this section we shall show a variant of [37, Corollary 3.8] which is based on the results in [12](see also [10]), [11](see also [14]), [6] and [7].

First, we shall define a homomorphism ρ from $F(A \otimes \mathcal{W})^\gamma$ to $\mathcal{B}^\gamma$. Let $\{k_n\}_{n=1}^\infty$ be an approximate unit for $\mathcal{W}$ with $k_{n+1} k_n = k_n$, and let $\mathcal{W}_0 := \{k_n b k_n \mid n \in \mathbb{N}, b \in \mathcal{W}\}$. Then $\mathcal{W}_0$ is a dense self-adjoint subalgebra of $\mathcal{W}$. For any $(x_n)_n \in (A \otimes \mathcal{W})_\omega$, $a \in A$, $b \in \mathcal{W}$ and $N \in \mathbb{N}$, we have

$$\begin{aligned}
((1_{A\sim} \otimes k_N) x_n (1_{A\sim} \otimes b)(a \otimes 1_{\mathcal{W}\sim}))_n &= ((1_{A\sim} \otimes k_N k_{N+1}) x_n (a \otimes b))_n \\
&= ((a \otimes k_N k_{N+1} b) x_n)_n \\
&= ((1_{A\sim} \otimes k_N) x_n (a \otimes k_{N+1} b))_n \\
&= ((a \otimes k_N k_{N+1}) x_n (1_{A\sim} \otimes b))_n \\
&= ((a \otimes 1_{\mathcal{W}\sim})(1_{A\sim} \otimes k_N) x_n (1_{A\sim} \otimes b))_n.
\end{aligned}$$

Hence $((1_{A\sim} \otimes k_N)x_n(1_{A\sim} \otimes b))_n \in \mathcal{A}$. For any $[(x_n)_n] \in F(A \otimes \mathcal{W})^\gamma$ and $k_N b k_N \in \mathcal{W}_0$, define

$$\rho([(x_n)_n] \otimes k_N b k_N) := [((1_{A\sim} \otimes k_N)x_n(1_{A\sim} \otimes b k_N))_n] \in \mathcal{B}.$$

We shall show this is well defined. Let $[(x_n)_n] = [(y_n)_n] \in F(A \otimes \mathcal{W})^\gamma$ and $k_N b k_N = k_{N'} b' k_{N'} \in \mathcal{W}_0$. For any $a \in A$, we have

$$(((1_{A\sim} \otimes k_N)x_n(1_{A\sim} \otimes b k_N) - (1_{A\sim} \otimes k_{N'})y_n(1_{A\sim} \otimes b' k_{N'}))(a \otimes 1_{\mathcal{W}\sim}))_n$$
$$= ((1_{A\sim} \otimes k_N)x_n(a \otimes b k_N) - (1_{A\sim} \otimes k_{N'})y_n(a \otimes b' k_{N'}))_n$$
$$= ((a \otimes k_N b k_N)x_n - (a \otimes k_{N'} b' k_{N'})y_n)_n = ((a \otimes k_N b k_N)(x_n - y_n))_n = 0.$$

Therefore $[((1_{A\sim} \otimes k_N)x_n(1_{A\sim} \otimes b k_N))_n] = [((1_{A\sim} \otimes k_{N'})y_n(1_{A\sim} \otimes b' k_{N'}))_n]$. By a similar argument, it can be easily checked that ρ is a homomorphism from the algebraic tensor product $F(A \otimes \mathcal{W})^\gamma \odot \mathcal{W}_0$ to $\mathcal{B}^\gamma$. Since we have

$$\begin{aligned}
\|\rho([(x_n)_n] \otimes k_N b k_N)\| &= \sup_{a \in A_{+,1}} \lim_{n \to \omega} \|(1_{A\sim} \otimes k_N)x_n(1_{A\sim} \otimes b k_N)(a \otimes 1_{\mathcal{W}\sim})\| \\
&= \sup_{a \in A_{+,1}} \lim_{n \to \omega} \|(a \otimes k_N b k_N)x_n\| \\
&\leq \sup_{a \in A_{+,1}} \lim_{n \to \omega} \|a\|\|k_N b k_N\|\|x_n\| \\
&= \lim_{n \to \omega} \|x_n\| \cdot \|k_N b k_N\|,
\end{aligned}$$

ρ can be extended to a homomorphism from the algebraic tensor product $F(A \otimes \mathcal{W})^\gamma \odot \mathcal{W}$ to $\mathcal{B}^\gamma$. Consequently, ρ can be extended to a homomorphism from $F(A \otimes \mathcal{W})^\gamma \otimes \mathcal{W}$ to $\mathcal{B}^\gamma$ because $\mathcal{W}$ is nuclear. By the construction of ρ, it is easy to show the following proposition.

Proposition 4.1. Let $(z_n)_n$ be an element in $\mathcal{A}$ such that $[(z_n)_n] = \rho([(x_n)_n] \otimes b)$ for some $[(x_n)_n] \in F(A \otimes \mathcal{W})^\gamma$ and $b \in \mathcal{W}$. Then

$$(z_n(a \otimes 1_{\mathcal{W}\sim}))_n = (x_n(a \otimes b))_n$$

for any $a \in A$.

Remark 4.2. Note that there exists an element $(x_n)_n$ in $(A \otimes \mathcal{W})_\omega$ such that $(x_n)_n \notin \mathcal{A}$. Indeed, if a is not an element in the center of A, $(a \otimes (k_n^2 - k_n))_n$ is such an element. But we do not know whether there exist $(x_n)_n \in (A \otimes \mathcal{W})_\omega$ and $b \in \mathcal{W}$ such that $(x_n(1_{A\sim} \otimes b))_n \notin \mathcal{A}$.

The following lemma is an analogous lemma of [37, Lemma 3.6].

Lemma 4.3. If x is a positive element in $F(A \otimes \mathcal{W})$, then

$$\tau_\mathcal{B}(\rho(x \otimes b)) = \tau_\omega(x)\tau_\mathcal{W}(b)$$

for any $b \in \mathcal{W}$.

Proof. Let $(z_n)_n$ be an element in $\mathcal{A}$ such that $[(z_n)_n] = \rho(x \otimes b)$, and let $\{h_n\}_{n=1}^\infty$ be an approximate unit for A. Note that $\tau_\mathcal{B}(\rho(x \otimes b)) = \lim_{n \to \omega} \tau(z_n)$. Since $\lim_{n \to \infty} \tau(h_n \otimes 1_\mathcal{W}) = 1$, a similar argument as in the proof of [35, Proposition 5.3] shows

$$\lim_{n \to \omega} \tau(z_n) = \lim_{m \to \infty} \lim_{n \to \omega} \tau(z_n(h_m \otimes 1_{\mathcal{W}\sim})).$$

By Proposition 4.1 and [37, Lemma 3.6],

$$\lim_{n \to \omega} \tau(z_n(h_m \otimes 1_{\mathcal{W}\sim})) = \tau_\omega(x)\tau(h_m \otimes b) = \tau_\omega(x)\tau_A(h_m)\tau_\mathcal{W}(b)$$

for any $m \in \mathbb{N}$. Therefore $\tau_\mathcal{B}(\rho(x \otimes b)) = \tau_\omega(x)\tau_\mathcal{W}(b)$ since $\lim_{m \to \infty} \tau_A(h_m) = 1$. $\square$

For a projection p in $F(A \otimes \mathcal{W})^\gamma$, let

$$\mathcal{B}_p^\gamma := \overline{\rho(p \otimes s)\mathcal{B}^\gamma \rho(p \otimes s)}$$

where s is a strictly positive element in $\mathcal{W}$. Note that $\mathcal{B}_p^\gamma$ is a hereditary subalgebra of $\mathcal{B}^\gamma$. Define a homomorphism σ_p from $\mathcal{W}$ to $\mathcal{B}_p^\gamma$ by

$$\sigma_p(b) = \rho(p \otimes b)$$

for any $b \in \mathcal{W}$.

Since the target algebra $\mathcal{B}^\gamma$ has strict comparison by Proposition 3.8, the same proof as [37, Proposition 3.7] shows the following proposition by using Lemma 4.3 instead of [37, Lemma 3.6]. See [37, Definition 3.2] for the definition of the (L, N)-fullness.

Proposition 4.4. There exist maps $L : \mathcal{W}_{+,1} \backslash \{0\} \times (0, 1) \to \mathbb{N}$ and $N : \mathcal{W}_{+,1} \backslash \{0\} \times (0, 1) \to (0, \infty)$ such that the following holds. If p be a projection in $F(A \otimes \mathcal{W})^\gamma$ such that $\tau_\omega(p) > 0$, then σ_p is (L, N)-full.

The following corollary is an immediate consequence of [37, Proposition 3.3] and the proposition above. For finite sets $\mathcal{F}_1$ and $\mathcal{F}_2$, let $\mathcal{F}_1 \odot \mathcal{F}_2 := \{a \otimes b \mid a \in \mathcal{F}_1, b \in \mathcal{F}_2\}$.

Corollary 4.5. Let Ω be a compact metrizable space. For any finite subsets $F_1 \subset C(\Omega)$, $F_2 \subset \mathcal{W}$ and $\varepsilon > 0$, there exist finite subsets $\mathcal{F}_1 \subset C(\Omega)$, $\mathcal{F}_2 \subset \mathcal{W}$, $m \in \mathbb{N}$ and $\delta > 0$ such that the following holds. Let p be a projection in $F(A \otimes \mathcal{W})^\gamma$ such that $\tau_\omega(p) > 0$. For any contractive $(\mathcal{F}_1 \odot \mathcal{F}_2, \delta)$-multiplicative maps $\varphi, \psi : C(\Omega) \otimes \mathcal{W} \to \mathcal{B}_p^\gamma$, there exist a unitary element u in $M_{m^2+1}(\mathcal{B}_p^\gamma)^\sim$ and $z_1, z_2, ..., z_m \in \Omega$ such that

$$\| u(\varphi(f \otimes b) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\rho(p \otimes b) \oplus \cdots \oplus \bigoplus_{k=1}^m f(z_k)\rho(p \otimes b)}^{m})u^*$$

$$- \psi(f \otimes b) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\rho(p \otimes b) \oplus \cdots \oplus \bigoplus_{k=1}^m f(z_k)\rho(p \otimes b)}^{m} \| < \varepsilon$$

for any $f \in F_1$ and $b \in F_2$.

5. Classification of normal elements in $F(A \otimes \mathcal{W})^\gamma$

In this section we shall classify certain normal elements in $F(A \otimes \mathcal{W})^\gamma$ up to unitary equivalence. Furthermore, we shall consider the comparison theory for certain projections in $F(A \otimes \mathcal{W})^\gamma$. We assume that Ω is a compact metrizable space in this section.

Using Proposition 2.6 and Proposition 2.7 instead of [37, Proposition 4.1], [37, Proposition 4.2] and [37, Proposition 4.8], we obtain the following lemma by the same proof as [37, Lemma 5.1]. See also [30, Lemma 4.1] and [30, Lemma 4.2].

Lemma 5.1. Let F be a finite subset of $C(\Omega)$ and $\varepsilon > 0$. Suppose that φ and ψ are unital homomorphisms from $C(\Omega)$ to $F(A \otimes \mathcal{W})^\gamma$ such that $\tau_\omega \circ \varphi = \tau_\omega \circ \psi$. Then there exist a projection $p \in F(A \otimes \mathcal{W})^\gamma$, (F, ε)-multiplicative unital c.p. maps φ' and ψ' from $C(\Omega)$ to $pF(A \otimes \mathcal{W})^\gamma p$, a unital homomorphism σ from $C(\Omega)$ to $(1 - p)F(A \otimes \mathcal{W})^\gamma(1 - p)$ with finite-dimensional range and a unitary element $u \in F(A \otimes \mathcal{W})^\gamma$ such that

$$0 < \tau_\omega(p) < \varepsilon, \ \|\varphi(f) - (\varphi'(f) + \sigma(f))\| < \varepsilon, \ \|\psi(f) - u(\psi'(f) + \sigma(f))u^*\| < \varepsilon$$

for any $f \in F$.

The following theorem is a variant of [37, Theorem 5.2]. See also [30, Theorem 4.5].

Theorem 5.2. Let F_1 be a finite subset of $C(\Omega)$, F_2 a finite subset of A and F_3 a finite subset of $\mathcal{W}$, and let $\varepsilon > 0$. Then there exist mutually orthogonal positive elements $h_1, h_2, ..., h_l$ in $C(\Omega)$ of norm one such that the following holds. For any $\nu > 0$, there exist finite subsets $\mathcal{G}_1 \subset C(\Omega)$, $\mathcal{G}_2 \subset A \otimes \mathcal{W}$ and $\delta > 0$ such that the following holds. If φ and ψ are unital c.p. maps from $C(\Omega)$ to $M(A \otimes \mathcal{W})$ such that

$$\tau(\varphi(h_i)) \geq \nu, \ \forall i \in \{1, 2, ..., l\},$$

$$\|[\varphi(f), x]\| < \delta, \ \|[\psi(f), x]\| < \delta, \ \forall f \in \mathcal{G}_1, x \in \mathcal{G}_2,$$

$$\|(\varphi(f_1 f_2) - \varphi(f_1)\varphi(f_2))x\| < \delta, \ \|(\psi(f_1 f_2) - \psi(f_1)\psi(f_2))x\| < \delta, \ \forall f_1, f_2 \in \mathcal{G}_1, x \in \mathcal{G}_2,$$

$$\|(\gamma_g(\varphi(f)) - \varphi(f))x\| < \delta, \ \|(\gamma_g(\psi(f)) - \psi(f))x\| < \delta, \ \forall g \in G, f \in \mathcal{G}_1, x \in \mathcal{G}_2,$$

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta, \ \forall f \in \mathcal{G}_1,$$

then there exists a contraction u in $(A \otimes \mathcal{W})^\sim$ such that

$$\|(a \otimes b)(u^* u - 1)\| < \varepsilon, \ \|(a \otimes b)(uu^* - 1)\| < \varepsilon, \ \|(a \otimes b)(\gamma_g(u) - u)\| < \varepsilon,$$

$$\|u\varphi(f)(a \otimes b)u^* - \psi(f)(a \otimes b)\| < \varepsilon$$

for any $f \in F_1$, $a \in F_2$, $b \in F_3$ and $g \in G$.

Proof. We may assume that every element in F_2 and F_3 is positive and of norm one. Take positive elements $h_1, h_2, ..., h_l$ in $C(\Omega)$ by the same way as in the proof of [37, Theorem 5.2]. We will show that $h_1, h_2, ..., h_l$ have the desired property. On the contrary, suppose that $h_1, h_2, ..., h_l$ did not have the desired property. Then there exists a positive number ν satisfying the following: For any $n \in \mathbb{N}$, there exist unital c.p. maps $\varphi_n, \psi_n : C(\Omega) \to M(A \otimes \mathcal{W})$ such that

$$\tau(\varphi_n(h_i)) \geq \nu, \ \forall i \in \{1, 2, ..., l\},$$

$$\|[\varphi_n(f_1), x]\| \to 0, \ \|[\psi_n(f_1), x]\| \to 0, \ \|(\varphi_n(f_1 f_2) - \varphi_n(f_1)\varphi_n(f_2))x\| \to 0,$$

$$\|(\psi_n(f_1 f_2) - \psi_n(f_1)\psi_n(f_2))x\| \to 0, \ \|(\gamma_g(\varphi_n(f_1)) - \varphi_n(f_1))x\| \to 0,$$

$$\|(\gamma_g(\psi_n(f_1)) - \psi_n(f_1))x\| \to 0, \ |\tau(\varphi_n(f_1)) - \tau(\psi_n(f_1))| \to 0$$

as $n \to \infty$ for any $f_1, f_2 \in C(\Omega)$, $x \in A \otimes \mathcal{W}$ and $g \in G$ and

$$\max_{f \in F_1, a \in F_2, b \in F_3} \|u\varphi_n(f)(a \otimes b)u^* - \psi_n(f)(a \otimes b)\| \geq \varepsilon$$

for any contraction u in $(A \otimes \mathcal{W})^\sim$ satisfying

$$\|(a \otimes b)(\gamma_g(u) - u)\| < \varepsilon, \ \|(a \otimes b)(u^* u - 1)\| < \varepsilon, \ \|(a \otimes b)(uu^* - 1)\| < \varepsilon$$

for any $a \in F_2$, $b \in F_3$ and $g \in G$.

Define homomorphisms φ and ψ from $C(\Omega)$ to $F(A \otimes \mathcal{W})^\gamma$ by $\varphi(f) := [(\varphi_n(f))_n]$ and $\psi(f) := [(\psi_n(f))_n]$ for any $f \in C(\Omega)$. Then we have

$$\tau_\omega \circ \varphi = \tau_\omega \circ \psi \quad \text{and} \quad \tau_\omega(\varphi(h_i)) \geq \nu$$

for any $i = 1, 2, ..., l$.

We obtain finite subsets $\mathcal{F}_1 \subset C(\Omega)$, $\mathcal{F}_2 \subset \mathcal{W}$, $m \in \mathbb{N}$ and $\delta > 0$ by applying Corollary 4.5 to F_1 and F_3 and $\varepsilon/7$. Put

$$F_1' := F_1 \cup \mathcal{F}_1 \cup \{h_1, h_2, ..., h_l\} \quad \text{and} \quad \varepsilon' := \min\left\{\frac{\varepsilon}{7}, \frac{\delta}{\max\{\|b\| \mid b \in \mathcal{F}_2\}}, \frac{\nu}{(m^2 + 2)}\right\}.$$

Applying Lemma 5.1 to F_1', ε', φ and ψ, there exist a projection $p \in F(A \otimes \mathcal{W})^\gamma$, (F_1', ε')-multiplicative unital c.p. maps φ' and ψ' from $C(\Omega)$ to $pF(A \otimes \mathcal{W})^\gamma p$, a unital homomorphism σ from $C(\Omega)$ to $(1 - p)F(A \otimes \mathcal{W})^\gamma (1 - p)$ with finite-dimensional range and a unitary element $w \in F(A \otimes \mathcal{W})^\gamma$ such that

$$0 < \tau_\omega(p) < \varepsilon', \ \|\varphi(f) - (\varphi'(f) + \sigma(f))\| < \varepsilon', \ \|\psi(f) - w(\psi'(f) + \sigma(f))w^*\| < \varepsilon'$$

for any $f \in F_1'$. The Choi-Effros lifting theorem implies that there exist sequences of contractive c.p. maps φ_n', ψ_n' and σ_n from $C(\Omega)$ to $A \otimes \mathcal{W}$ such that $\varphi'(f) = [(\varphi_n'(f))_n]$, $\psi'(f) = [(\psi_n'(f))_n]$ and $\sigma(f) = [(\sigma_n(f))_n]$ for any $f \in C(\Omega)$. By [37, Proposition 4.9], there exists a unitary element $(w_n)_n$ in $(A \otimes \mathcal{W})_\omega^\sim$ such that $w = [(w_n)_n]$. Note that we have $(x\gamma_g(w_n))_n = (xw_n)_n$ for any $g \in G$ and $x \in A \otimes \mathcal{W}$,

$$\lim_{n \to \omega} \|\varphi_n(f)(a \otimes b) - (\varphi_n'(f) + \sigma_n(f))(a \otimes b)\| < \frac{\varepsilon}{7} \tag{1}$$

and

$$\lim_{n \to \omega} \|\psi_n(f)(a \otimes b) - w_n(\psi_n'(f) + \sigma_n(f))(a \otimes b)w_n^*\| < \frac{\varepsilon}{7} \tag{2}$$

for any $f \in F_1'$, $a \in F_2$ and $b \in F_3$.

Define c.p. maps Φ' and Ψ' from $C(\Omega) \otimes \mathcal{W}$ to $\mathcal{B}_p^\gamma$ by

$$\Phi' := \rho \circ (\varphi' \otimes \mathrm{id}_\mathcal{W}) \quad \text{and} \quad \Psi' := \rho \circ (\psi' \otimes \mathrm{id}_\mathcal{W}).$$

Then Φ' and Ψ' are contractive $(\mathcal{F}_1 \odot \mathcal{F}_2, \delta)$-multiplicative maps. Hence Corollary 4.5 implies that there exist a unitary element U in $M_{m^2+1}(\mathcal{B}_p^\gamma)^\sim$ and $z_1, z_2, ..., z_m \in \Omega$ such that

$$\|U(\Phi'(f \otimes b) \oplus \overbrace{\bigoplus_{k=1}^{m} f(z_k)\rho(p \otimes b) \oplus \cdots \oplus \bigoplus_{k=1}^{m} f(z_k)\rho(p \otimes b)}^{m})U^*$$

$$- \Psi'(f \otimes b) \oplus \overbrace{\bigoplus_{k=1}^{m} f(z_k)\rho(p \otimes b) \oplus \cdots \oplus \bigoplus_{k=1}^{m} f(z_k)\rho(p \otimes b)}^{m} \| < \frac{\varepsilon}{7}$$

for any $f \in F_1$ and $b \in F_3$.

Using Proposition 2.3 instead of [37, Proposition 4.1], the same argument as in the proof of [37, Theorem 5.2] shows that there exist mutually orthogonal projections $\{p_{j,k}\}_{j,k=1}^{m}$ in $(1-p)F(A \otimes \mathcal{W})^\gamma(1-p)$ and a homomorphism $\sigma'' : C(\Omega) \to (1-p-q)F(A \otimes \mathcal{W})^\gamma(1-p-q)$ where $q = \sum_{j,k=1}^{m} p_{j,k}$ such that

$$\|\sigma(f) - \left(\sum_{j=1}^{m} \sum_{k=1}^{m} f(z_k)p_{j,k} + \sigma''(f) \right)\| < \frac{2\varepsilon}{7}$$

for any $f \in F_1$ and $p_{j,k}$ is Murray-von Neumann equivalent to p for any $j, k = 1, 2, ..., m$. Define a homomorphism $\hat{\sigma}$ from $C(\Omega)$ to $F(A \otimes \mathcal{W})^\gamma$ by

$$\hat{\sigma}(f) := \sum_{j=1}^{m} \sum_{k=1}^{m} f(z_k)p_{j,k} + \sigma''(f)$$

for any $f \in C(\Omega)$. By the Choi-Effros lifting theorem, there exists a sequence of contractive c.p. maps $\hat{\sigma}_n$ from $C(\Omega)$ to $A \otimes \mathcal{W}$ such that $\hat{\sigma}(f) = [(\hat{\sigma}_n(f))_n]$. Note that we have

$$\lim_{n \to \omega} \|\sigma_n(f)(a \otimes b) - \hat{\sigma}_n(f)(a \otimes b)\| < \frac{2\varepsilon}{7} \tag{3}$$

for any $f \in F_1$, $a \in F_2$ and $b \in F_3$.

Since we can regard $\Phi'(f \otimes b) + \sum_{j=1}^{m} \sum_{k=1}^{m} f(z_k)\rho(p_{j,k} \otimes b) \in \mathcal{B}_{p+q}^\gamma$ as an element in $M_{m^2+1}(\mathcal{B}_p^\gamma)$, the same argument as in the proof of [37, Theorem 5.2] shows that there exists a unitary element V in $(\mathcal{B}^\gamma)^\sim$ such that

$$\|V(\Phi'(f \otimes b) + \rho(\hat{\sigma}(f) \otimes b))V^* - (\Psi'(f \otimes b) + \rho(\hat{\sigma}(f) \otimes b))\| < \frac{\varepsilon}{7}$$

for any $f \in F_1$ and $b \in F_3$. Let $(v_n)_n$ be a contraction in $\mathcal{A}^\sim$ such that $V = [(v_n)_n]$. Then we have $((a \otimes 1_{\mathcal{W}^\sim})v_n^*v_n)_n = ((a \otimes 1_{\mathcal{W}^\sim})v_nv_n^*)_n = a \otimes 1_{\mathcal{W}^\sim}$ and

$((a \otimes 1_{\mathcal{W}\sim})\gamma_g(v_n))_n = ((a \otimes 1_{\mathcal{W}\sim})v_n)_n$ for any $g \in G$ and $a \in A$. Furthermore, we see that

$$\lim_{n \to \omega} \|v_n(\varphi_n'(f) + \hat{\sigma}_n(f))(a \otimes b)v_n^* - (\psi_n'(f) + \hat{\sigma}_n(f))(a \otimes b)\| < \frac{\varepsilon}{7} \qquad (4)$$

for any $f \in F_1$, $a \in F_2$ and $b \in F_3$ by Proposition 4.1.

Put $(u_n)_n := (w_n v_n)_n \in ((A \otimes \mathcal{W})^\sim)^\omega$. Then we have

$$((a \otimes b)u_n^* u_n)_n = ((a \otimes b)v_n^* v_n)_n = a \otimes b$$

and

$$((a \otimes b)u_n u_n^*)_n = (w_n(a \otimes b)v_n v_n^* w_n^*)_n = (w_n(a \otimes b)w_n^*)_n = a \otimes b$$

for any $a \in A$ and $b \in \mathcal{W}$. Also, we have

$$((a \otimes b)\gamma_g(u_n))_n = ((a \otimes b)\gamma_g(w_n)\gamma_g(v_n))_n = ((a \otimes b)w_n \gamma_g(v_n))_n$$
$$= (w_n(a \otimes b)\gamma_g(v_n))_n = (w_n(a \otimes b)v_n)_n = ((a \otimes b)u_n)_n$$

for any $g \in G$, $a \in A$ and $b \in \mathcal{W}$. By (1), (2), (3) and (4), we see that

$$\lim_{n \to \omega} \|u_n \varphi_n(f)(a \otimes b)u_n^* - \psi_n(f)(a \otimes b)\| < \varepsilon$$

for any $f \in F_1$, $a \in F_2$ and $b \in F_3$. Therefore, taking a sufficiently large n, we obtain a contradiction. Consequently, the proof is complete. $\square$

The following theorem is the main result in this section.

Theorem 5.3. Let N_1 and N_2 be normal elements in $F(A \otimes \mathcal{W})^\gamma$ such that $\mathrm{Sp}(N_1) = \mathrm{Sp}(N_2)$ and $\tau_\omega(f(N_1)) > 0$ for any $f \in C(\mathrm{Sp}(N_1))_+ \setminus \{0\}$. Then there exists a unitary element u in $F(A \otimes \mathcal{W})^\gamma$ such that $uN_1 u^* = N_2$ if and only if $\tau_\omega(f(N_1)) = \tau_\omega(f(N_2))$ for any $f \in C(\mathrm{Sp}(N_1))$.

Proof. By a similar argument as in the proof of [37, Theorem 5.3], we can prove this theorem. We shall give a proof for reader's convenience.

Since the only if part is clear, we will show the if part. Define unital homomorphisms φ and ψ from $C(\mathrm{Sp}(N_1))$ to $F(A \otimes \mathcal{W})^\gamma$ by $\varphi(f) := f(N_1)$ and $\psi(f) := f(N_2)$, respectively. By the Choi-Effros lifting theorem, we see that there exist sequences of unital c.p. maps φ_n and ψ_n from $C(\mathrm{Sp}(N_1))$ to $(A \otimes \mathcal{W})^\sim$ such that $f(N_1) = [(\varphi_n(f))_n]$ and $f(N_2) = [(\psi_n(f))_n]$ for any $f \in C(\mathrm{Sp}(N_1))$. Then we have

$$|\tau(\varphi_n(f_1)) - \tau_\omega(f_1(N_1))| \to 0, \ \|[\varphi_n(f_1), x]\| \to 0, \ \|[\psi_n(f_1), x]\| \to 0,$$

$$\|(\varphi_n(f_1 f_2) - \varphi_n(f_1)\varphi_n(f_2))x\| \to 0, \ \|(\psi_n(f_1 f_2) - \psi_n(f_1)\psi_n(f_2))x\| \to 0,$$

$$\|(\gamma_g(\varphi_n(f_1)) - \varphi_n(f_1))x\| \to 0, \ \|(\gamma_g(\psi_n(f_1)) - \psi_n(f_1))x\| \to 0,$$

$$|\tau(\varphi_n(f_1)) - \tau(\psi_n(f_1))| \to 0$$

as $n \to \omega$ for any $f_1, f_2 \in C(\mathrm{Sp}(N_1))$, $x \in A \otimes \mathcal{W}$ and $g \in G$.

We denote by ι the identity function on $\mathrm{Sp}(N_1)$, that is, $\iota(z) = z$ for any $z \in \mathrm{Sp}(N_1)$. Let $F_1 := \{1, \iota\} \subset C(\mathrm{Sp}(N_1))$, and let $\{F_{2,k}\}_{k \in \mathbb{N}}$ and $\{F_{3,k}\}_{k \in \mathbb{N}}$ be increasing sequences of finite subsets in A and $\mathcal{W}$ such that $A = \overline{\bigcup_{k \in \mathbb{N}} F_{2,k}}$ and $\mathcal{W} = \overline{\bigcup_{k \in \mathbb{N}} F_{3,k}}$, respectively. For any $k \in \mathbb{N}$, we obtain mutually orthogonal positive elements $h_{1,k}, h_{2,k}, ..., h_{l(k),k}$ in $C(\mathrm{Sp}(N_1))$ of norm one by applying Theorem 5.2 to F_1, $F_{2,k}$, $F_{3,k}$ and $1/k$. Put

$$\nu_k := \frac{1}{2} \min\{\tau_\omega(h_{1,k}(N_1)), \tau_\omega(h_{2,k}(N_1)), ..., \tau_\omega(h_{l(k),k}(N_1))\} > 0.$$

Applying Theorem 5.2 to ν_k, we obtain finite subsets $\mathcal{G}_{1,k} \subset C(\mathrm{Sp}(N_1))$, $\mathcal{G}_{2,k} \subset A \otimes \mathcal{W}$ and $\delta_k > 0$. We may assume that $\{\mathcal{G}_{1,k}\}_{k \in \mathbb{N}}$ and $\{\mathcal{G}_{2,k}\}_{k \in \mathbb{N}}$ are increasing

sequences and $\delta_k > \delta_{k+1}$ for any $k \in \mathbb{N}$. We can find a sequence $\{X_k\}_{k=1}^{\infty}$ of elements in ω such that $X_k \subset X_{k+1}$ and for any $n \in X_k$,

$$|\tau(\varphi_n(h_{i,k})) - \tau_\omega(h_{i,k}(N_1))| < \nu_k, \; \|[\varphi_n(f_1), x]\| < \delta_k, \; \|[\psi_n(f_1), x]\| < \delta_k,$$

$$\|(\varphi_n(f_1 f_2) - \varphi_n(f_1)\varphi_n(f_2))x\| < \delta_k, \; \|(\psi_n(f_1 f_2) - \psi_n(f_1)\psi_n(f_2))x\| < \delta_k,$$

$$\|(\gamma_g(\varphi_n(f_1)) - \varphi_n(f_1))x\| < \delta_k, \; \|(\gamma_g(\psi_n(f_1)) - \psi_n(f_1))x\| < \delta_k,$$

$$|\tau(\varphi_n(f_1)) - \tau(\psi_n(f_1))| < \delta_k$$

for any $i \in \{1, 2, ..., l(k)\}$, $f_1, f_2 \in \mathcal{G}_{1,k}$, $x \in \mathcal{G}_{2,k}$ and $g \in G$. Since we have

$$\tau(\varphi_n(h_{i,k})) > \tau_\omega(h_{i,k}(N_1)) - \nu_k \geq 2\nu_k - \nu_k = \nu_k$$

for any $i \in \{1, 2, ..., l(k)\}$, Theorem 5.2 implies that for any $n \in X_k$, there exists a contraction $u_{k,n}$ in $(A \otimes \mathcal{W})^\sim$ such that

$$\|(a \otimes b)(u_{k,n}^* u_{k,n} - 1)\| < \frac{1}{k}, \; \|(a \otimes b)(u_{k,n} u_{k,n}^* - 1)\| < \frac{1}{k},$$

$$\|(a \otimes b)(\gamma_g(u_{k,n}) - u_{k,n})\| < \frac{1}{k}, \; \|u_{k,n}\varphi_n(f)(a \otimes b)u_{k,n}^* - \psi_n(f)(a \otimes b)\| < \frac{1}{k}$$

for any $f \in F_1$, $a \in F_{2,k}$, $b \in F_{3,k}$ and $g \in G$. Since $F_1 = \{1, \iota\}$, we have

$$\|[u_{k,n}, a \otimes b]\| \leq \|u_{k,n}(a \otimes b)(1 - u_{k,n}^* u_{k,n})\| + \|(u_{k,n}(a \otimes b)u_{k,n}^* - a \otimes b)u_{k,n}\| < \frac{2}{k}$$

and

$$\|u_{k,n}\varphi_n(\iota)(a \otimes b)u_{k,n}^* - \psi_n(\iota)(a \otimes b)\| < \frac{1}{k}$$

for any $n \in X_k$, $a \in F_{2,k}$ and $b \in F_{3,k}$. Put

$$u_n := \begin{cases} 1 & \text{if } n \notin X_1 \\ u_{k,n} & \text{if } n \in X_k \setminus X_{k+1} \quad (k \in \mathbb{N}) \end{cases}.$$

Then

$$\|(a \otimes b)(u_n^* u_n - 1)\| \to 0, \; \|(a \otimes b)(u_n u_n^* - 1)\| \to 0, \; \|(a \otimes b)(\gamma_g(u_n) - u_n)\| \to 0,$$

$$\|[u_n, a \otimes b]\| \to 0, \; \|(u_n\varphi_n(\iota)u_n^* - \psi_n(\iota))(a \otimes b)\| \to 0$$

as $n \to \omega$ for any $a \in A$, $b \in \mathcal{W}$ and $g \in G$. Therefore $[(u_n)_n]$ is a unitary element in $F(A \otimes \mathcal{W})^\gamma$ and $[(u_n)_u]N_1[(u_n)_n]^* = N_2$. $\square$

Applying the theorem above to projections, we obtain the following corollary. Note that if p is a projection, then $C(\mathrm{Sp}(p))$ can be identified with $\{\lambda_1 p + \lambda_2(1 - p) \mid \lambda_1, \lambda_2 \in \mathbb{C}\}$. Hence it is clear that $\tau_\omega(f(p)) > 0$ for any $f \in C(\mathrm{Sp}(p))_+ \setminus \{0\}$ if and only if $0 < \tau_\omega(p) < 1$. Also, for projections p and q, we have $\tau_\omega(f(p)) = \tau_\omega(f(q))$ for any $f \in C(\mathrm{Sp}(p))$ if and only if $\tau_\omega(p) = \tau_\omega(q)$.

Corollary 5.4. Let p and q be projections in $F(A \otimes \mathcal{W})^\gamma$ such that $0 < \tau_\omega(p) < 1$. Then p and q are unitarily equivalent if and only if $\tau_\omega(p) = \tau_\omega(q)$.

The following corollary is important in the next section.

Corollary 5.5. Let p and q be projections in $F(A \otimes \mathcal{W})^\gamma$ such that $0 < \tau_\omega(p) \leq 1$. Then p and q are Murray-von Neumann equivalent if and only if $\tau_\omega(p) = \tau_\omega(q)$.

Proof. By Corollary 5.4, it suffices to show that if p is a projection in $F(A \otimes \mathcal{W})^\gamma$ such that $\tau_\omega(p) = 1$, then p is Murray-von Neumann equivalent to 1. Proposition 2.6 implies that there exists a projection r in $F(A \otimes \mathcal{W})^\gamma$ such that $r \leq p$ and $\tau_\omega(r) = 1/2$. By Corollary 5.4, $p - r$ is unitarily equivalent to $1 - r$. Therefore $p = (p - r) + r$ is Murray-von Neumann equivalent to $(1 - r) + r = 1$. $\square$

6. Rohlin type theorem

In this section we shall show that γ has the Rohlin property.

For a γ-cocycle w in $F(A \otimes \mathcal{W})$, define an action γ^w on $F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C})$ by

$$\gamma_g^w := \mathrm{Ad}\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & w(g) \end{array} \right)\right) \circ (\gamma_g \otimes \mathrm{id})$$

for any $g \in G$. Since γ has the weak Rohlin property, we obtain the following lemma by similar arguments as in [33, Proposition 4.8] and [34, Proposition 3.3] (see also arguments in Section 3). We leave the proof to the reader.

Lemma 6.1. Let a and b be positive elements in $(F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C}))^{\gamma^w}$ such that $d_{\tau_\omega \otimes \mathrm{Tr}_2}(a) < d_{\tau_\omega \otimes \mathrm{Tr}_2}(b)$ where Tr_2 is the (unnormalized) usual trace on $M_2(\mathbb{C})$. Then there exists an element r in $(F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C}))^{\gamma^w}$ such that $r^*br = a$.

The proof of the following lemma is based on Connes' 2×2 matrix trick in [5, Corollary 2.6].

Lemma 6.2. Every γ-cocycle w in $F(A \otimes \mathcal{W})$ is a coboundary.

Proof. Let $\varepsilon > 0$. By Proposition 2.6, there exists a projection p_ε in $F(A \otimes \mathcal{W})^\gamma$ such that $\tau_\omega(p_\varepsilon) = 1 - \varepsilon$. Taking a suitable subsequence of a representative of p_ε, we may assume that $w(g)p_\varepsilon = p_\varepsilon w(g)$ for any $g \in G$. Lemma 6.1 implies that there exists an element R_ε in $(F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C}))^{\gamma^w}$ such that

$$R_\varepsilon^* \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) R_\varepsilon = \left(\begin{array}{cc} 0 & 0 \\ 0 & p_\varepsilon \end{array} \right).$$

The diagonal argument shows that there exist a projection p in $F(A \otimes \mathcal{W})^\gamma$ and an element R in $(F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C}))^{\gamma^w}$ such that $\tau_\omega(p) = 1$ and

$$R^* \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) R = \left(\begin{array}{cc} 0 & 0 \\ 0 & p \end{array} \right).$$

By Corollary 5.5, there exists an element s in $F(A \otimes \mathcal{W})^\gamma$ such that $s^*s = 1$ and $ss^* = p$. Taking suitable subsequences of representatives of s, p and R, we may assume that $w(g)s = sw(g)$ for any $g \in G$ and

$$\left(\begin{array}{cc} 0 & 0 \\ 0 & s^* \end{array} \right) R^* \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) R \left(\begin{array}{cc} 0 & 0 \\ 0 & s \end{array} \right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).$$

It it easy to see that there exists a projection q in $F(A \otimes \mathcal{W})^\gamma$ such that $\tau_\omega(q) = 1$ and

$$\left(\begin{array}{cc} q & 0 \\ 0 & 0 \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) R \left(\begin{array}{cc} 0 & 0 \\ 0 & p \end{array} \right) R^* \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).$$

By Corollary 5.5, there exists an element t in $F(A \otimes \mathcal{W})^\gamma$ such that $t^*t = 1$ and $tt^* = q$. Put

$$V := \left(\begin{array}{cc} 0 & 0 \\ 0 & s^* \end{array} \right) R^* \left(\begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right).$$

Then we have $V \in (F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C}))^{\gamma^w}$,

$$V^*V = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \quad \text{and} \quad VV^* = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).$$

It is easy to see that there exists a unitary element v in $F(A \otimes \mathcal{W})$ such that

$$V = \left(\begin{array}{cc} 0 & 0 \\ v & 0 \end{array} \right).$$

Since $V \in (F(A \otimes \mathcal{W}) \otimes M_2(\mathbb{C}))^{\gamma^w}$, $w(g)\gamma_g(v) = v$ for any $g \in G$. Consequently, w is a coboundary. $\qquad\square$

Remark 6.3. The lemma above shows that the first cohomology of γ vanishes. This property is one of the important properties for the Bratteli-Elliott-Evans-Kishimoto intertwining argument (see, for example, [13] and [27]) in the classification of Rohlin actions.

The following theorem is the main result in this paper.

Theorem 6.4. Let A be a simple separable nuclear C*-algebra with a unique tracial state and no unbounded traces, and let α be a strongly outer action of a finite group G on A. Then $\gamma = \alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{W}$ has the Rohlin property.

Proof. We identify $B(\ell^2(G))$ with $M_{|G|}(\mathbb{C})$. Also, we can identify $F(A \otimes \mathcal{W})^\gamma$ with $F(A \otimes \mathcal{W} \otimes \bigotimes_{n \in \mathbb{N}} M_{|G|}(\mathbb{C}))^{\gamma \otimes \mathrm{id}}$ because $\mathcal{W}$ is UHF stable. Let λ be the left regular representation of G on $\ell^2(G)$. Define a map w from G to $F(A \otimes \mathcal{W})^\gamma$ by

$$w(g) := [(h_n \otimes k_n \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n} \otimes \lambda(g) \otimes 1 \otimes \cdots)_n]$$

where $\{h_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ are approximate units for A and $\mathcal{W}$, respectively. Then w is a homomorphism, and hence w is a γ-cocycle in $F(A \otimes \mathcal{W})$. By Lemma 6.2, there exists a unitary element v in $F(A \otimes \mathcal{W})$ such that $w(g) = v\gamma_g(v^*)$ for any $g \in G$. For any $g \in G$, let e_g be a projection onto $\mathbb{C}\delta_g$ where $\{\delta_h \mid h \in G\}$ is the canonical basis of $\ell^2(G)$, and put

$$p_g := v^*[(h_n \otimes k_n \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n} \otimes e_g \otimes 1 \otimes \cdots)_n]v.$$

Then $\{p_g\}_{g \in G}$ is a partition of unity in $F(A \otimes \mathcal{W})$ consisting of projections satisfying

$$\gamma_g(p_h) = p_{gh}$$

for any $g, h \in G$. Consequently, γ has the Rohlin property. $\qquad \square$

Combining the theorem above and the classification results in [2] and [10], we obtain the following corollary.

Corollary 6.5. Let A and B be simple separable nuclear C*-algebras with a unique tracial state and no unbounded traces, and let α and β be strongly outer actions of a finite group G on A and B, respectively. Then $\alpha \otimes \mathrm{id}$ on $A \otimes \mathcal{W}$ is conjugate to $\beta \otimes \mathrm{id}$ on $B \otimes \mathcal{W}$.

Proof. By [2, Theorem 6.1], $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$ have finite nuclear dimension. Hence [10, Corollary 6.7] implies that $A \otimes \mathcal{W}$ and $B \otimes \mathcal{W}$ are isomorphic to $\mathcal{W}$. Therefore we obtain the conclusion by Theorem 2.5 and Theorem 6.4. $\qquad \square$

Remark 6.6. (1) If α' is not a strongly outer action of a non-trivial finite group G on A, then $(A \otimes \mathcal{W}) \rtimes_{\alpha' \otimes \mathrm{id}} G$ has at least two extremal tracial state. Hence $\alpha' \otimes \mathrm{id}$ is not (cocycle) conjugate to the action in the corollary above.
(2) There exist uncountably many non-conjugate strongly outer actions of $\mathbb{Z}_2$ on $\mathcal{W}$ by [36, Example 5.6] and [36, Remark 5.7].
(3) For generalizing the corollary above to amenable group actions, it seems to be important that we characterize $\mathcal{W}$ by using the central sequence C*-algebra $F(\mathcal{W})$.

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