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Contents

1 Basics of functional analysis

Definition 1.1. A *semi-norm* on a vector space X over $\mathbb{C}$ is a function $p\colon X \to [0,\infty)$ such that for $\xi, \eta \in X$ and $\alpha \in \mathbb{C}$,

(1) $p(\xi+\eta) \le p(\xi) + p(\eta)$

(2) $p(\alpha\xi) = |\alpha| p(\xi)$.

A *norm* is a semi-norm $\|\cdot\|$ satisfying

$$\|\xi\| = 0 \iff \xi = 0.$$

Remark 1.2. If X has a norm, then $d(\xi,\eta) = \|\xi - \eta\|$ defines a metric on X.

Definition 1.3. A *Banach space* is a complete normed space.

Definition 1.4. A *semi-inner product* on a vector space X over $\mathbb{C}$ is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ such that for $\xi,\eta,\zeta \in X$ and $\alpha \in \mathbb{C}$,

(1) $\langle \xi+\eta, \zeta\rangle = \langle \xi,\zeta\rangle + \langle \eta,\zeta\rangle$,

(2) $\langle \alpha\xi, \eta\rangle = \alpha\langle\xi,\eta\rangle$,

(3) $\langle\xi,\eta\rangle = \overline{\langle\eta,\xi\rangle}$,

(4) $\langle\xi,\xi\rangle \ge 0$.

A *inner product* is a semi-inner product satisfying

$$\langle\xi,\xi\rangle = 0 \iff \xi = 0.$$

Remark 1.5. If X has an (semi-)inner product, then $p(\xi) = \langle\xi,\xi\rangle^{1/2}$ defines a (semi-)norm on X.

Theorem 1.6 (Cauchy-Bunyakowsky-Schwarz inequality). If $\langle\cdot,\cdot\rangle$ is a semi-inner product on X, then

$$|\langle\xi,\eta\rangle| \le \|\xi\|\|\eta\|.$$

Proof. 省略. □

Definition 1.7. A *Hilbert space* $\mathcal{H}$ is a Banach space with respect to $\|\xi\| := \langle\xi,\xi\rangle^{1/2}$.

A set $\{\xi_i\}$ of vectors is *orthonormal* if $\langle\xi_i,\xi_j\rangle = \delta_{ij}$. A *maximal* orthonormal set is an *orthonormal basis.*

Proposition 1.8. If $\{\xi_i\}$ is an ONS in $\mathcal{H}$, then there is an ONB in $\mathcal{H}$, which contains $\{\xi_i\}$.

Proof. Use Zorn's lemma. □

Theorem 1.9. Let $\{\xi_i\}$ be an ONS in $\mathcal{H}$. Then the following are equivalent:

(1) $\{\xi_i\}$ is an ONB in $\mathcal{H}$,

(2) For any $\xi \in \mathcal{H}$, $\xi = \sum_i \langle\xi,\xi_i\rangle\xi_i$, (**Fourier series expansion**)

(3) For any $\xi \in \mathcal{H}$, $\|\xi\|^2 = \sum_i |\langle\xi,\xi_i\rangle|^2$, (**Riesz-Fischer identity**)

(4) For any $\xi,\eta \in \mathcal{H}$, $\langle\xi,\eta\rangle = \sum_i \langle\xi,\xi_i\rangle\langle\xi_i,\eta\rangle$, (**Paseval identity**)

Remark 1.10. If $\mathcal{H}$ is a separable Hilbert space, then there is a countable ONB $\{\xi_n\}$ in $\mathcal{H}$.

Example 1.11. Let S be a countable set. Then

$$\ell_2 S := \{f\colon S \to \mathbb{C} \mid \sum_{s\in S} |f(s)|^2 < \infty\}$$

is a Hilbert space with an inner product

$$\langle f, g\rangle := \sum_{s\in S} f(s)\overline{g(s)}.$$

If $\delta_s(t) = \delta_{s,t}$ (Kronecker delta), then a set $\{\delta_s\}_{s\in S}$ is the canonical ONB for $\ell_2 S$. If $|S| = n < \infty$, then $\ell_2 S = \mathbb{C}^n$.

Definition 1.12. Let X be a normed space. A linear functional $f\colon X \to \mathbb{C}$ is *bounded* if

$$\|f\| := \sup_{\|\xi\|\le 1} |f(\xi)| < \infty.$$

We denote by X^* the *dual space* of X, i.e., the set of all bounded linear functionals on X, which becomes a Banach space.

Theorem 1.13 (Hahn-Banach extension theorem). Let Y be a subspace of a normed space X. Then

(1) For any $g \in Y^*$, there is $f \in X^*$ such that $f|_Y = g$ and $\|g\| = \|f\|$.

(2) For any $0 \neq \xi \in X$, there is $f \in X^*$ such that $f(\xi) = \|\xi\|$ and $\|f\| = 1$.

Proof. Use Zorn's lemma. □

Definition 1.14. Let X be a normed space. Then $X^{**} := (X^*)^*$ is a Banach space, which is called the *second dual space* of X.

For $x \in X$, we define $\hat{x} \in X^{**}$ by $\hat{x}(f) := f(x)$ for $f \in X^*$. Note that $\|x\| = \|\hat{x}\|$.

Definition 1.15. Let X be a normed space. For any $f \in X^*$, we define a semi-norm p_f on X by $p_f(\xi) := |f(\xi)|$ for $\xi \in X$. The *weak topology* on X is defined by the family $\{p_f\}_{f\in X^*}$ of semi-norms. Hence X becomes a *locally convex topological vector space.*

For any $\xi \in X$, we define a semi-norm p_ξ on X^* by $p_\xi(f) := |f(\xi)|$ for $f \in X^*$. The *weak-$*$ topology* on X^* is defined by the family $\{p_\xi\}_{\xi\in X}$ of semi-norms. Hence X^* becomes a locally convex topological vector space.

We remark that we can also define the weak topology in X^*, which is coming from X^{**}.

Theorem 1.16. Let C be a convex subset of a normed space X. Then C is norm closed if and only if it is weakly closed.

Proof. Use Hahn-Banach separation theorem. □

Theorem 1.17 (Banach-Alaoglu). If X is a normed space, then $(X^*)_1 := \{f \in X^*\colon \|f\| \le 1\}$ is compact in X^* in the weak-$*$ topology.

Example 1.18. Let S be a countable set with $|S| = \infty$. For $1 \le p < \infty$, we define a Banach space

$$\ell_p S := \{f \colon \Gamma \to \mathbb{C} \mid \|f\|_p := \left(\sum_{s\in S} |f(s)|^p\right)^{1/p} < \infty\}.$$

For $p = \infty$, we define a Banach space

$$\ell_\infty S := \{f \colon \Gamma \to \mathbb{C} \colon \|f\|_\infty := \sup_{s\in S} |f(s)| < \infty\}.$$

We also define

$$c_c S := \{f \in \ell_\infty S \mid |\mathrm{supp}(f)| < \infty\}$$

and

$$c_0 S := \{f \in \ell_\infty S \mid \lim_{s\to\infty} f(s) = 0\} = \overline{c_c S}^{\|\cdot\|_\infty}.$$

Then we have $c_c S \subset \ell_p S \subset c_0 S \subset \ell_\infty S$. Moreover, if $1 \le q < p \le \infty$, then for $f \in \ell_q S$ we have $\|f\|_p \le \|f\|_q$, and thus $\ell_q S \subset \ell_p S$. Note that $c_c S$ is also dense in $\ell_p S$ with respect to the norm $\|\cdot\|_p$ for $1 \le p < \infty$.

For $1 \le p, q \le \infty$ with $1/p + 1/q = 1$, we have

(1) $\|fg\|_1 \le \|f\|_p \|g\|_q$ (**Hölder inequality**)

(2) $\|f + g\|_p \le \|f\|_p + \|g\|_p$ (**Minkowski inequality**)

If $1 \le p < \infty$, $1 < q \le \infty$ with $1/p + 1/q = 1$, then $(\ell_p S)^* = \ell_q S$ via the identification

$$\ell_q S \ni g \mapsto \hat{g} \in (\ell_p S)^*,$$

where

$$\hat{g}(f) := \sum_{s\in S} f(s)g(s). \quad (f \in \ell_p S)$$

We remark that $(\ell_\infty S)^* \ne \ell_1 S$ and $(c_0 S)^* = \ell_1 S$.

Example 1.19. Let X be a compact Hausdorff space. We denote by $C(X)$ the set of all $\mathbb{C}$-valued continuous functions on X. Then $C(X)$ is a Banach space with respect to a norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

We denote by $M(X)$ the set of all regular $\mathbb{C}$-valued Borel measures on X, and $M(X)_+$ the set of all regular finite positive Borel measures on X. Note that $M(X) = \mathrm{span}\, M(X)_+$.

Theorem 1.20 (Riesz-Markov-Kakutani). For $\varphi \in C(X)^*$ with $\varphi \ge 0$, there is unique $\mu \in M(X)_+$ such that

$$\varphi(f) = \int_X f d\mu \quad (f \in C(X)).$$

It follows that $C(X)^* = M(X)$.

Theorem 1.21 (Stone-Weierstrass). If a subalgebra A of $C(X)$ satisfies

(1) for any $x \ne x'$, there is $f \in A$ such that $f(x) \ne f(x')$,

(2) if $f \in A$, then $\overline{f} \in A$,

(3) $1 \in A$,

then A is dense in $C(X)$.

2 Basics of C*-algebras

Definition 2.1. An *algebra* is a vector space A over $\mathbb{C}$ with a multiplication : $A \times A \ni (a, b) \mapsto ab \in A$ satisfying the following conditions: for any $a, b, c \in A$ and $\alpha \in \mathbb{C}$,

(1) $(ab)c = a(bc)$,

(2) $(\alpha a)b = a(\alpha b) = \alpha(ab)$,

(3) $a(b + c) = ab + ac$,

(4) $(a + b)c = ac + bc$.

If A is an algebra, then we say A is *abelian* if $ab = ba$ for any $a, b \in A$. We also say A is *unital* if there exists the unit $1 \in A$ such that $1a = a1 = a$ for any $a \in A$.

A *Banach algebra* is a complete normed algebra A with a norm satisfying the following conditions:

$$\|ab\| \leq \|a\|\|b\| \text{ for any } a, b \in A.$$

If A is a Banach algebra and A has a unit with $\|1\| = 1$, then A is called a unital Banach algebra.

A $*$-*algebra* is an algebra A with a involution $A \ni a \mapsto a^* \in A$ satisfying the following conditions: for any $a, b \in A$ and $\alpha \in \mathbb{C}$,

(1) $(a^*)^* = a$,

(2) $(a + b)^* = a^* + b^*$,

(3) $(\alpha a)^* = \overline{\alpha}a$,

(4) $(ab)^* = b^*a^*$.

A *C^*-algebra* A is a Banach algebra with a involution satisfying the so-called **C*-condition**:

$$\|a^*a\| = \|a\|^2 \text{ for } a \in A.$$

Remark 2.2. If A is a C^*-algebra, then $\|a^*\| = \|a\|$ for any $a \in A$. Moreover, if A is unital, then $1^* = 1$ and $\|1\| = 1$. **[Problem 1]**

Example 2.3. Let S be a countable set with $|S| = \infty$. Then $\ell_\infty S$ is a unital C^*-algebra. **[Problem 2]**

Example 2.4. Let X be a compact Hausdorff space. Then $C(X)$ is a unital C^*-algebra. **[Problem 3]**

Example 2.5. Let $\mathcal{H}$ be a (separable) Hilbert space. Then $\mathbb{B}(\mathcal{H})$ is a unital C^*-algebra. If $\dim \mathcal{H} = \infty$, then $\mathbb{K}(\mathcal{H})$ is non-unital. More generaly, norm-closed $*$-subalgebra $A \subset \mathbb{B}(\mathcal{H})$ is a (concrete) C^*-algebra.

If $\dim \mathcal{H} = n < \infty$, then $\mathcal{H} = \mathbb{C}^n$ and $\mathbb{B}(\mathcal{H}) = \mathbb{M}_n$ (the set of all $n \times n$ matrices over $\mathbb{C}$).

Definition 2.6. Let A be unital Banach algebra and $a \in A$. We say a is *invertible* in A if there exists $b \in A$ such that $ba = ab = 1$. Notice that such b is unique, and so we may write $a^{-1} := b$. The set

$$GL(A) := \{a \in A \mid a \text{ is invertible in } A\}.$$

is a group under the multiplication.

Definition 2.7. Let A be unital Banach algebra and $a \in A$. We define the spectrum of a by

$$\sigma(a) = \sigma_A(a) := \{\alpha \in \mathbb{C} \mid a - \alpha 1 \notin GL(A)\},$$

Example 2.8. If $f \in C(X)$, then $\sigma(f) = f(X)$. **[Problem 4]**

Example 2.9. If $T \in \mathbb{M}_n$, then $\sigma(T)$ is the set of all eigenvalues of T.

Theorem 2.10. Let A be unital Banach algebra and $a \in A$. Then $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$.

Proof. 省略. (複素解析を使う.) □

Theorem 2.11 (Gelfand-Mazur). Let A be a unital Banach algebra. If any non-zero $a \in A$ is invertible, then $A = \mathbb{C}$.

Proof. Let $0 \neq a \in A$. Then there is $\alpha \in \sigma(a)$. Hence $a - \alpha 1$ is not invertible and must be zero, i.e., $a = \alpha 1$. □

Definition 2.12. Let A be unital Banach algebra and $a \in A$. We define the *spectral radius* of a by

$$r(a) := \{|\alpha| \mid \alpha \in \sigma(a)\}.$$

Example 2.13. If $f \in C(X)$, then $r(f) = \|f\|_\infty$.

Example 2.14. If

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2,$$

then $\|a\| = 1$, but $r(a) = 0$. **[Problem 5]**

Theorem 2.15 (Beurling). Let A be a unital Banach algebra and $a \in A$. Then

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}.$$

Proof. 省略. □

Corollary 2.16. Let A be a unital C^*-algebra. If $a \in A$ is norml, i.e., $a^*a = aa^*$, then $\|a\| = r(a)$.

Proof. Since

$$\|a^{2^n}\|^2 = \|(a^{2^n})^* a^{2^n}\| = \|(a^*a)^{2^n}\| = \|(a^*a)^{2^{n-1}}\|^2 = \cdots = \|a^*a\|^{2^n} = \|a\|^{2^{n+1}},$$

we have $\|a\| = \|a^{2^n}\|^{1/2^n} \to r(a)$. □

Remark 2.17. If A is a unital C^*-algebra, then $\|a\| = \|a^*a\|^{1/2} = r(a^*a)^{1/2}$. Hence the C^*-norm is completely determined by its algebraic structure and it is unique.

Definition 2.18. Let A be a unital (Banach) algebra. A subspace I of A is said to be *left* (resp. *right*) *ideal of* A if

$$a \in A \text{ and } b \in I \Longrightarrow ab \in I \text{ (resp. } ba \in I).$$

An *ideal* in A is a left and a right ideal in A.

Example 2.19. Let Y be a closed subset of a compact Hausdorff space X. Then

$$I_Y := \{f \in C(X) \mid f|_Y = 0\}$$

is an ideal of $C(X)$. [**Problem 6**]

Example 2.20. The matrix algebra $\mathbb{M}_n$ has no proper ideal. [**Problem 7**]

Proposition 2.21. Let A be a unital Banach algebra, $I \subset A$ a closed ideal. Then the quotient space A/I becomes a unital Banach algebra as follows:

(1) $[a] + [b] := [a + b]$,

(2) $\alpha[a] := [\alpha a]$,

(3) $[a][b] := [ab]$,

(4) $\|[a]\| := \inf\{\|a + b\| \colon b \in I\}$,

where $[a] := a + I = \{a + b \mid b \in I\} \in A/I$.

Proof. 省略. □

Remark 2.22. What is the quotient algebra $C(X)/I_Y$ for a closed subset Y of X. [**Problem 8**]

Definition 2.23. A *maxiaml* ideal in a unital (Banach) algebra A is a proper ideal in A, which is not contained in any other proper ideal in A.

Example 2.24. For any element $x \in X$, then $I_{\{x\}}$ is a maximal ideal in $C(X)$. [**Problem 9**]

Remark 2.25. For any ideal I of unital (Banach) algebra A, by Zorn's lemma, there exists a maximal ideal J of A such that $I \subset J$.

Proposition 2.26. Let I be an ideal of unital Banach algebra A. Then

(1) The closure $\overline{I}$ is an ideal of A.

(2) If I is maximal, then I is closed.

Proof. 省略. □

Theorem 2.27. Let I be an ideal of unital abelian Banach algebra A. Then I is maximal if and only if $A/I = \mathbb{C}$.

Proof. An ideal I is maximal if and only if A/I is a field. Use Gelfand-Mazur theorem. □

Definition 2.28. Let A, B be (unital) algebras. A *homomorphism* from A to B is a linear map $\pi \colon A \to B$ such that $\pi(ab) = \pi(a)\pi(b)$ for any $a, b \in A$. If $\pi(1_A) = 1_B$, then we say π is *unital*. When A, B are $*$-algebras, we say π is $*$-*homomorphism* if $\pi(a^*) = \pi(a)^*$.

A *character* on an abelian algebra A is a non-zero homomorphism $\chi \colon A \to \mathbb{C}$. We denote by $\hat{A}$ the set of all characters on A.

Example 2.29. For $x \in X$, we define a character χ_x on $C(X)$ by $\chi_x(f) := f(x)$ for $f \in C(X)$. Then $\ker \chi_x = I_{\{x\}}$, i.e., it is a maximal ideal.

Theorem 2.30. Let A be a unital abelian Banach algebra.

(1) If $\chi \in \hat{A}$, then $\chi(1) = 1$ and $\|\chi(a)\| \le \|a\|$.

(2) $\hat{A} \neq \emptyset$ and the map $\chi \mapsto \ker \chi$ is a bijection from $\hat{A}$ onto the set of all maximal ideals of A.

(3) $\sigma(a) = \{\chi(a) \colon \chi \in \hat{A}\}$ for $a \in A$.

Proof. (1) It is easy to see that $\chi(1) = 1$. Hence $\|\chi\| \ge 1$. Suppose that $\|\chi\| > 1$, i.e., there is $0 \neq a \in A$ such that $\|a\| < 1 = \chi(a)$. If we put $b = \sum_{n \in \mathbb{N}} a^n \in A$, then $a + ab = b$. Therefore we have

$$\chi(b) = \chi(a) + \chi(a)\chi(b) = 1 + \chi(b),$$

which is a contradiction.

(2) It is easy to show that $\ker \chi$ is a maximal ideal for any $\chi \in \hat{A}$. Conversely, if $I \subset A$ is a maximal ideal, then $A/I = \mathbb{C}$. Hence we define a character $\chi \colon A \ni a \mapsto [a] \in A/I = \mathbb{C}$, which satisfies $\ker \chi = I$.

(3) If $\alpha \in \sigma(a)$, then $a - \alpha 1$ is not invertible. Hence there is a maximal ideal $I = \ker \chi$ such that $a - \alpha 1 \in I$. So $\chi(a) = \alpha$. Conversely, if $\alpha = \chi(a)$, then $\chi(a - \alpha 1) = 0$. Hence $a - \alpha 1$ is not invertible. □

Theorem 2.31. Let A be a unital abelian Banach algebra. Then $\hat{A} \subset A^*$ is a weak-$*$ compact Hausdorff space.

Proof. It is easy to see that $\hat{A}$ is weak-$*$ closed. By Banach-Alaoglu theorem, it is weak-$*$ compact. □

Definition 2.32. Let A be a unital abelian Banach algebra. For $a \in A$, we define $\hat{a} \in C(\hat{A})$ by $\hat{a}(\chi) = \chi(a)$. Then we define the *Gelfand transform* $\gamma \colon A \to C(\hat{A})$ by $\gamma(a) = \hat{a}$.

Theorem 2.33 (Gelfand-Naimark). Let A be a unital abelian Banach algebra. The the Gelfand transform γ is a norm-decreasing homomorphism and $\|\hat{a}\|_\infty = r(a)$ for $a \in A$.

If A is C^*-algebra, then γ is isometric $*$-isomorphism.

Proof. It is easy to see that γ is homomorphism. For any $a \in A$, we have

$$\|\gamma(a)\| = \|\hat{a}\|_\infty = \sup_{\chi \in \hat{A}} |\hat{a}(\chi)| = r(a) \le \|a\|.$$

Now assume that A is a C^*-algebra. Since A is abelian, any $a \in A$ is normal. Hence $\|\hat{a}\|_\infty = r(a) = \|a\|$ and so γ is isometric.

It is easy to check that $\gamma(A) \subset C(\hat{A})$ is closed $*$-subalgebra. By Stone-Weierstrass theorem, we have $\gamma(A) = C(\hat{A})$. □

Definition 2.34. Let A be a unital C^*-algebra and $a \in A$. We say

(1) a is *unitay* if $a^*a = aa^* = 1$,

(2) a is *self-adjoint* if $a^* = a$.

We denote by $\mathcal{U}(A)$ the set of all unitaries in A, and by A_{sa} the set of all self-adjoint elements in A.

Theorem 2.35. Let A be a unital C^*-algebra and $a \in A$. Then

(1) If a is unitary, then $\sigma(a) \subset \mathbb{T} = \{\alpha \in \mathbb{C} \colon |\alpha| = 1\}$.

(2) If a is self-adjoint, then $\sigma(a) \subset [-\|a\|, \|a\|]$.

Proof. 省略. □

Theorem 2.36. Let B be a unital C^*-subalgebra of a unital C^*-algebra A with $1_B = 1_A$. Then $\sigma_B(a) = \sigma_A(a)$ for $a \in B$.

Proof. It is trivial that $\sigma_A(a) \subset \sigma_B(a)$. Conversely, let $b = a - \alpha 1 \in B$. Then it suffices to show that if $\exists b^{-1} \in A$, then $b^{-1} \in B$. If b is self-adjoint, then $\sigma_B(b) \subset \mathbb{R}$. Hence for any $\varepsilon > 0$, we have $(b - i\varepsilon 1)^{-1} \in B$. Since $\|(b - i\varepsilon 1)^{-1} - b^{-1}\| \to 0$, we have $b^{-1} \in B$. If b is not self-adjoint, then $(b^*b)^{-1} \in A$ implies $(b^*b)^{-1} \in B$. Hence $b^{-1} = (b^*b)^{-1}b^* \in B$. □

Definition 2.37. Let A be a unital C^*-algebra and $a \in A$ normal. We denote by $C^*(a)$ a unital abelian C^*-subalgebra of A, which is generated by a.

Theorem 2.38. Let A be a unital C^*-algebra and $a \in A$ normal. The map $\hat{a} \colon \widehat{C^*(a)} \ni \chi \mapsto \chi(a) \in \sigma(a)$ is homeomorphic. Hence it induces the isometric $*$-isomorphism $\gamma^{-1} \circ \hat{a}^t \colon C(\sigma(a)) \to C^*(a)$ with $z \mapsto a$, where z is the inclusion map of $\sigma(a)$ in $\mathbb{C}$.

Proof. 省略. □

Definition 2.39. For a normal element a in a unital C^*-algebra A, we denote by γ_a the unique unital $*$-homomorphism from $C(\sigma(a))$ to A, which is called the *functional calculus* of a. If p is a polynomial, then $\gamma_a(p) = p(a)$, so for $f \in C(\sigma(a))$ we write $f(a) = \gamma_a(f)$.

Theorem 2.40 (Spectral Mapping). Let A be a unital C^*-algebra and $a \in A$ normal. Then $\sigma(f(a)) = f(\sigma(a))$ for $f \in C(\sigma(a))$.

Proof. 省略. □

Definition 2.41. Let A be a unital C^*-algebra. We say $a \in A$ is *positive* if a is self-adjoint and $\sigma(a) \subset [0, \infty)$. In this case, we write $a \geq 0$. We also denote $A_+ = \{a \in A \colon a \geq 0\}$.

Definition 2.42. For self-adjoint elements a, b in a unital C^*-algebra A, we write $a \leq b$ if $b - a \geq 0$.

Example 2.43. If $A = C(X)$, then $f \in C(X)$ is positive if and only if $f(x) \geq 0$ for any $x \in X$.

Theorem 2.44. Let A be a unital C^*-algebra and $a \in A$. Then $a \geq 0$ if and only if $a = b^*b$ for some $b \in A$.

Proof. If $a \geq 0$, then there is $b \geq 0$ such that $a = b^2$. Conversely, if $a = b^*b$, then a is self-adjoint. Moreover there are $a_+, a_- \geq 0$ such that $a = a_+ - a_-$ and $a_+a_- = 0$. Hence it suffices to show that $a_- = 0$. If we set $c = ba_-$, then $c^*c = a_-b^*ba_- = -a_-^3 \leq 0$. Since $\sigma(c^*c) \cup \{0\} = \sigma(cc^*) \cup \{0\}$, $cc^* \leq 0$. Since $c^*c = 2\mathrm{Re}\,(c)^2 + 2\mathrm{Im}\,(c)^2 - cc^* \geq 0$, $c^*c \in A_+ \cap (-A_+) = \{0\}$. Hence $a_- = 0$. □

Theorem 2.45. Let A be a unital C^*-algebra and $a, b, c \in A$.

(1) $a \geq b \geq 0 \Longrightarrow \|a\| \geq \|b\|$,

(2) $a \geq b \Longrightarrow c^*ac \geq c^*bc$,

(3) a, b are invertible and $a \geq b \geq 0 \Longrightarrow 0 \leq b^{-1} \leq a^{-1}$.

Proof. (1) Use the Gelfand transform.

(2) Use the previous theorem.

(3) First prove that if $c \geq 1$, then $c^{-1} \leq 1$, by using the Gelfand transform. Next put $c = a^{-1/2} b a^{-1/2} \geq 1$. □

Theorem 2.46. Let A, B be unital C^*-algebras and $\pi\colon A \to B$ a unital $*$-homomorphism. Then

(1) $\pi(a) \geq 0$ for $a \in A_+$,

(2) $\|\pi(a)\| \leq \|a\|$ for $a \in A$,

(3) If π is injective, then π is isometric.

Proof. (1) Easy.

(2) Since $\sigma(\pi(a)) \subset \sigma(a)$, we have

$$\|a\|^2 = \|a^*a\| = r(a^*a) \geq r(\pi(a^*a)) = r(\pi(a)^*\pi(a)) = \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.$$

(3) It suffices to show that $\|\pi(a^*a)\| = \|a^*a\|$. Hence we may assume that A, B are abelian. We define $\pi'\colon \hat{B} \ni \chi \mapsto \chi \circ \pi \in \hat{A}$. Then we have $\pi'(\hat{B}) = \hat{A}$. Hence for $a \in A$,

$$\|a\| = \|\hat{a}\|_\infty = \sup_{\chi \in \hat{A}} |\chi(a)| = \sup_{\chi \in \hat{B}} |\chi(\pi(a))| = \|\pi(a)\|.$$

□

Definition 2.47. Let A be a unital C^*-algebra. A linear functional $\omega\colon A \to \mathbb{C}$ is *positive* if $\omega(a) \geq 0$ for $a \in A_+$.

Example 2.48. Any positive linear functional ω on $C(X)$ is given by $\mu \in M(X)_+$ via

$$\omega(f) = \int_X f(x) d\mu(x).$$

(**Riesz-Markov-Kakutani representation theorem**.)

Example 2.49. Any positive linear functional ω on $\mathbb{M}_n$ is given by $h \in \mathbb{M}_{n,+}$ such that

$$\omega(a) = \mathrm{Tr}(ah),$$

where Tr is the canonical trace on $\mathbb{M}_n$.

Proposition 2.50 (Schwarz inequality). If ω is a positive linear functional on a unital C^*-algebra A, then

$$|\omega(b^*a)|^2 \leq \omega(b^*b)\omega(a^*a)$$

for any $a, b \in A$.

Proof. Notice that $\langle a, b\rangle = \omega(b^*a)$ is a semi-inner product on A. □

Theorem 2.51. Let A be a unital C^*-algebra. If ω is a positive linear functional on a unital C^*-algebra, then ω is bounded with $\|\omega\| = \omega(1)$.

Proof. If $\|a\| \leq 1$, then $0 \leq a^*a \leq 1$. Hence by Schwarz inequality,

$$|\omega(a)|^2 = |\omega(1a)|^2 \leq \omega(1)\omega(a^*a) \leq \omega(1)^2.$$

□

Theorem 2.52. Let A be a unital C^*-algebra and $\omega \in A^*$. Then ω is positive if and only if $\omega(1) = \|\omega\|$.

Proof. Suppose that $\omega(1) = \|\omega\| = 1$. First show that $\omega(a) \in \mathbb{R}$ for $a \in A_{\text{sa}}$. Next if $a \geq 0$ with $\|a\| = 1$, then $1 - a \in A_{\text{sa}}$ and $\|1 - a\| \leq 1$. So $1 - \omega(a) = \omega(1 - a) \leq 1$. □

Definition 2.53. Let A be a unital C^*-algebra. We denote by A^*_+ the set of all positive linear functionals on A. If $\omega \in A^*_+$ with $\|\omega\| = \omega(1) = 1$, then we call it a *state*. We denote by $S(A)$ the set of all states on A.

Theorem 2.54. Let A be a unital C^*-algebra. Then $S(A)$ is a weak-$*$ compact convex subset of A^*.

Proof. Since $S(A) = \{\omega \in A^*_+ \colon \omega(1) = 1\}$, it is weak-$*$ closed convex. By Bnach-Alaoglu theorem, $S(A)$ is weak-$*$ compact. □

Theorem 2.55. Let A be a non-zero unital C^*-algebra and $a \in A$ normal. Then there is $\omega \in S(A)$ such that $\omega(a) = \|a\|$.

Proof. We may assume that $a \neq 0$. Since $B = C^*(a)$ is abelian, there is $\chi \in \hat{B}$ such that $\|a\| = \|\hat{a}\|_\infty = |\chi(a)|$. By Hahn-Banach extension theorem, there is an extension ω such that $\|\omega\| = 1$. Since $\omega(1) = \chi(1) = 1$, ω is positive with $\|\omega\| = 1$. □

Definition 2.56. Let A be a unital C^*-algebra and $\omega \in S(A)$. Then

$$N_\omega := \{a \in A \colon \omega(a^*a) = 0\}$$

is a closed left ideal of A. (Use Schwarz inequality.) Next we define a inner product on A/N_ω by

$$\langle [a], [b] \rangle := \omega(b^*a),$$

and denote by $\mathcal{H}_\omega$ the completion of A/N_ω. Now we define a $*$-homomorphism $\pi_\omega \colon A \to \mathbb{B}(\mathcal{H}_\omega)$ by

$$\pi_\omega(a)[b] := [ab].$$

If we set $\xi_\omega = [1] \in \mathcal{H}_\omega$, then ξ_ω is *cyclic* for π_ω, i.e., $\pi_\omega(A)\xi_\omega$ is dense in $\mathcal{H}_\omega$. We say $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ is the *GNS repersentation* associated with ω.

Theorem 2.57 (Gelfand-Naimark). If A is a unital C^*-algebra, then it has a faithful representation.

Proof. We define the *universal* representation $\pi_u := \bigoplus_{\omega \in S(A)} \pi_\omega$. If $\pi_u(a) = 0$, then $\pi_\omega(a^*a) = 0$ for any $\omega \in S(A)$. If we put $b = (a^*a)^{1/4}$, then $\|\pi_u(b)\|^4 = \|\pi_u(b)^4\| = \|\pi_u(a^*a)\| = 0$ and so $\pi_u(b) = 0$. Therefore there is $\omega \in S(A)$ such that $\|a^*a\| = \omega(a^*a) = \omega(b^4) = \|\pi_\omega(b)[b]\|^2$=0. Hence $a = 0$. □

3 “Classical” group C^*-algebras

Definition 3.1. Let Γ be a countable discrete group. Then $c_c\Gamma$ becomes a unital $*$-algebra with the multiplication

$$f * g(s) := \sum_{t \in \Gamma} f(t) g(t^{-1}s)$$

and the involution

$$f^*(s) := \overline{f(s^{-1})}$$

with the unit δ_e. The above operations can be also defined on $\ell_1\Gamma$, which becomes a unital $*$-algebra.

Definition 3.2. A *unitay representation* of Γ is a homomorphism of Γ into the unitary group of $\mathbb{B}(\ell_2\Gamma)$. We denote by λ the *left regular representation*:

$$(\lambda(s)f)(t) := f(s^{-1}t) \quad (s, t \in \Gamma).$$

Remark 3.3. Let $\{\delta_t\}_{t\in\Gamma}$ be the canonical ONB for $\ell_2\Gamma$. Then

$$\lambda(s)\delta_t = \delta_{st} \quad (s, t \in \Gamma).$$

[Problem 10]

Lemma 3.4. There is a one-to-one correspondence between the set of all unitary representation of Γ and the set of all representations of $c_c\Gamma$ (or $\ell_1\Gamma$):

$$\pi \mapsto \tilde{\pi}(f) := \sum_{s\in\Gamma} f(s)\pi(s), \quad (f \in c_c\Gamma)$$

and

$$\|\tilde{\pi}(f)\| \le \|f\|_1.$$

Proof. 省略. □

Remark 3.5. For $f \in c_c\Gamma$, we have

$$\tilde{\lambda}(f)g = f * g \quad (g \in \ell_2\Gamma).$$

[Problem 11]

We also simply write π for the extened representation $\tilde{\pi}$ of $c_c\Gamma$.

Lemma 3.6. The extended representation λ of $c_c\Gamma$ (or $\ell_1\Gamma$) is injective.

Proof. 省略. □

Definition 3.7. The *reduced group C^*-algebra* is defined to be $C^*_\lambda\Gamma := \overline{\lambda(c_c\Gamma)} = \overline{\lambda(\ell_1\Gamma)} \subset \mathbb{B}(\ell_2\Gamma)$.

The *full group C^*-algebra* it the completion of $c_c\Gamma$ with respect to the C^*-norm

$$\|f\|_u := \sup\{\|\pi(f)\| \colon \pi \text{ is a unitary representation of } \Gamma\}.$$

Example 3.8. Let $\Gamma = \mathbb{Z} = \langle a \rangle$ be the integer group. The Fourier transform induces the unitary $u\colon \ell_2\mathbb{Z} \to L^2(\mathbb{T}), f \mapsto \mathcal{F}(f) = \hat{f}$, which is defined by

$$\hat{f}(z) := \sum_{n\in\mathbb{Z}} f(n)z^n.$$

Then for any $f \in c_c\mathbb{Z}$ and $g \in \ell_2\mathbb{Z}$, we have

$$u\lambda(f)u^*\hat{g} = u\lambda(f)g = \mathcal{F}(f * g) = \hat{f}\hat{g} = M_{\hat{f}}\hat{g},$$

where $M_f \in \mathbb{B}(L^2(\mathbb{T}))$ is defined by $M_f g := fg$ for $f \in C(\mathbb{T})$ and $g \in L^2(\mathbb{T})$, which gives an isometric $*$-homomorphism $C(\mathbb{T}) \to \mathbb{B}(L^2(\mathbb{T}))$. Hence the map $\lambda(f) \mapsto u\lambda(f)u^* = M_{\hat{f}}$ gives a isometric $*$-isomorphism between $C^*_\lambda\mathbb{Z}$ and $C(\mathbb{T})$.

Since $\mathbb{Z}$ is abelian, $C^*\mathbb{Z}$ is a unital abelian C^*-algebra. By the Gelfand transform, we have $C^*\mathbb{Z} = C(\widehat{C^*\mathbb{Z}})$. For each chracter χ on $C^*\mathbb{Z}$, we have a scalar $z = \chi(\delta_a) \in \mathbb{T}$ and this gives a homeomorphism. Therefore $C^*\mathbb{Z} = C^*_\lambda\mathbb{Z} = C(\mathbb{T})$.

More generally, for every abelian group Γ, the *Pontryagin duality* gives $C^*\Gamma = C^*_\lambda\Gamma = C(\hat{\Gamma})$.

Proposition 3.9. Let $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ be a unitary representation. Then there is a unique $*$-homomorphism $\overline{\pi}\colon C^*(\Gamma) \to \mathbb{B}(\mathcal{H})$ such that $\overline{\pi}(f) = \pi(f)$ for $f \in c_c\Gamma$.

Proof. It follows from $\|\pi(f)\| \leq \|f\|_u$ for $f \in c_c\Gamma$. □

Definition 3.10. A function $\varphi\colon \Gamma \to \mathbb{C}$ is said to be *positive definite* if the matrix

$$[\varphi(s^{-1}t)]_{s,t\in F} \in \mathbb{M}_F$$

is positive for any finite subset $F \subset \Gamma$, i.e.,

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\varphi(s_i^{-1}s_j)\alpha_j \geq 0$$

for any $n \in \mathbb{N}$, $s_1, \ldots, s_n \in \Gamma$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

We denote by $P(\Gamma)$ the set of all positive definite functions on Γ.

Example 3.11. For $f \in c_c\Gamma$, the function $f^* * f$ is positive definite. [**Problem 12**]

Remark 3.12. Let $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ be a unitary representation and $\xi \in \mathcal{H}$. If we define

$$\varphi(s) := \langle \pi(s)\xi, \xi\rangle,$$

then φ is positive definite. [**Problem 13**]

Proposition 3.13. Let $f \in c_c\Gamma$. Then the following are equivalent:

(1) f is positive definite,

(2) $\lambda(f)$ is positive.

Proof. For a finite subset $F \subset \Gamma$, set $\xi = \sum_{s\in F} \alpha_s\delta_s \in \ell_2\Gamma$. Then

$$\langle \lambda(f)\xi, \xi\rangle = \sum_{r\in \mathrm{supp}(f)} \sum_{s,t\in F} f(r)\alpha_s\overline{\alpha_t}\langle \lambda(r)\delta_s, \delta_t\rangle = \sum_{s,t\in F} \overline{\alpha_t} f(ts^{-1})\alpha_s.$$

□

Definition 3.14. For a function $\varphi\colon \Gamma \to \mathbb{C}$, we define a correspoinding functional $\omega_\varphi\colon c_c\Gamma \to \mathbb{C}$ by

$$\omega_\varphi(f) = \sum_{s\in\Gamma} f(s)\varphi(s).$$

Theorem 3.15. Let φ be function with $\varphi(e) = 1$. The following are equivalent:

(1) φ is positive definite.

(2) there exists a unitary representation λ_φ of Γ on a Hilbert space $\mathcal{H}_\varphi$ and a cyclic vector ξ_φ such that

$$\varphi(s) = \langle \lambda_\varphi(s)\xi_\varphi, \xi_\varphi\rangle.$$

(3) ω_φ extends to a state on $C^*\Gamma$.

Proof. (1)$\Longrightarrow$(2): Let φ be a positive definite function. Define a semi-inner product on $c_c\Gamma$ by

$$\langle f, g\rangle_\varphi = \sum_{s,t\in\Gamma} \varphi(s^{-1}t)f(t)\overline{g(s)}.$$

By the separation and the completion, we get a Hilbert space $\ell_2^\varphi\Gamma$. Then we define $\lambda_\varphi(s)[f] = [sf]$ for $f \in c_c\Gamma$ and $\xi_\varphi = [\delta_e]$, which satisfy desired properties, where $(sf)(t) = f(s^{-1}t)$.

(2)$\Longrightarrow$(3): Trivial.

(3)$\Longrightarrow$(1): If we write

$$f = \sum_{i=1}^n \alpha_i\delta_{s_i} \in c_c\Gamma,$$

then

$$\sum_{i,j=1}^n \overline{\alpha_i}\varphi(s_i^{-1}s_j)\alpha_j = \omega_\varphi(f^* * f) \geq 0.$$

□

Corollary 3.16. The map $P(\Gamma) \ni \varphi \mapsto \omega_\varphi \in (C^*\Gamma)^*_+$ gives a bijection.

Proposition 3.17. Let φ_1, φ_2 be positive definite functions on Γ. Then the product $\varphi_1\varphi_2$ is also positive definite.

Proof. Let $a_k = [a_{ij}^{(k)}]$, $a_{ij}^{(k)} = \varphi_k(s_i^{-1}s_j)$ for $k = 1, 2$. Then a_1, a_2 are positive matrices. Then $a = a_1 \circ a_2 = [a_{ij}^{(1)}a_{ij}^{(2)}]$ (*Schur product*) is also positive. Hence if $\xi = [\alpha_1, \ldots, \alpha_n] \in \mathbb{C}^n$, then

$$\sum_{i,j=1}^n \overline{\alpha_i}\varphi_1(s_i^{-1}s_j)\varphi_2(s_i^{-1}s_j)\alpha_j = \langle a\xi, \xi\rangle \geq 0.$$

□

Definition 3.18. A group Γ is *amenable* if there exists a state $\mu \in \ell_\infty\Gamma$ which is invariant under left translation: for any $s \in \Gamma$ and $f \in \ell_\infty\Gamma$, $\mu(sf) = \mu(f)$.

Definition 3.19. Let $\mathrm{Prob}(\Gamma)$ be the space of all probability measures on Γ:

$$\mathrm{Prob}(\Gamma) = \{\mu \in \ell_1\Gamma \colon \mu \geq 0, \sum_{s\in\Gamma} \mu(s) = 1\}.$$

Definition 3.20. We say Γ has an *approximate invariant mean* if for any finite subset $F \subset \Gamma$ and $\varepsilon > 0$, there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\max_{s\in E} \|s\mu - \mu\|_1 < \varepsilon,$$

where $s\mu(F) = \mu(s^{-1}F)$ for $F \subset \Gamma$.

Definition 3.21. We say Γ satisfies the *Følner condition* if for any finite subset $E \subset \Gamma$ and $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that

$$\max_{s\in E} \frac{|sF \triangle F|}{|F|} < \varepsilon,$$

where $sF = \{st \colon t \in F\}$.

Example 3.22. All abelian groups are amenable by the **Markov-Kakutani fixed point theorem**.

Example 3.23. The free group $\mathbb{F}_d$ is not amenable for $d \geq 2$. Let $d = 2$ and a, b be the free generators. Set

$$A^+ = \{\text{all reduced words starting with } a\} \subset \mathbb{F}_d,$$

similarly let A^-, B^+, B^-. Then for $C = \{e, b, b^2, \ldots, \} \subset \mathbb{F}_d$, we have

$$\begin{aligned} \mathbb{F}_d &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \sqcup C) \\ &= A^+ \sqcup aA^- \\ &= b^{-1}(B^+ \setminus C) \sqcup (B^- \sqcup C). \end{aligned}$$

Suppose that there is an invariant state μ on $\ell_\infty \mathbb{F}_d$. Then

$$\begin{aligned} 1 = \mu(1) &= \mu(\chi_{A^+}) + \mu(\chi_{A^-}) + \mu(\chi_{B^+ \setminus C}) + \mu(\chi_{B^- \sqcup C}) \\ &= \mu(\chi_{A^+}) + \mu(a\chi_{A^-}) + \mu(b^{-1}\chi_{B^+ \setminus C}) + \mu(\chi_{B^- \sqcup C}) \\ &= 2\mu(1) = 2, \end{aligned}$$

which is a contradiction.

More generally, if Γ contains $\mathbb{F}_d$, then Γ is non-amenable.

Theorem 3.24. Let Γ be a contable discrete group. Then the following are equivalent:

(1) Γ is amenable,

(2) Γ has an approximate invariant mean,

(3) Γ satisfies the Følner condition,

(4) there is unit vectors $\xi_i \in \ell_2\Gamma$ such that $\|\lambda(s)\xi_i - \xi_i\|_2 \to 0$ for $s \in \Gamma$,

(5) there is a sequence (φ_i) of finitly supported positive definite functions on Γ such that $\varphi_i(s) \to 1$ for $s \in \Gamma$,

(6) $C^*\Gamma = C^*_\lambda\Gamma$,

(7) $C^*_\lambda\Gamma$ has a character, i.e., one-dimenssional representation.

Proof. (1)$\Longrightarrow$(2): Let μ be an invariant mean on $\ell_\infty\Gamma$. Since $\ell_1\Gamma$ is weak-$*$ dence in $(\ell_\infty\Gamma)^*$, there is a sequence $\mu_i \in \text{Prob}(\Gamma)$ such that $\mu_i \to \mu$ in $(\ell_\infty\Gamma)^*$ in the weak-$*$ topology. Since $(\ell_1\Gamma)^* = \ell_\infty\Gamma$, we have $s\mu_i - \mu_i \to 0$ in $\ell_1\Gamma$ in the weak topology. Hence for any $s_1, \ldots, s_n \in \Gamma$, since the weak and norm closed doincide on a convex subset, we have

$$0 \in \overline{\text{conv}} \bigoplus_{i=1}^n \{s_i\mu - \mu \colon \mu \in \text{Prob}(\Gamma)\} \subset (\ell_1\Gamma)^n.$$

(2)$\Longrightarrow$(3): Let $E \subset \Gamma$ be a finite subset and $\varepsilon > 0$. Choose $\mu \in \text{Prob}(\Gamma)$ such that

$$\sum_{s \in E} \|s\mu - \mu\|_1 < \varepsilon.$$

For $f \in \ell_1\Gamma$ with $f \geq 0$ and $r \geq 0$, we define

$$F(f, r) = \{t \in \Gamma \colon f(t) > r\}.$$

Observe that if $f(t) > g(t)$, then

$$|\chi_{F(f,r)}(t) - \chi_{F(g,r)}(t)| = 1 \iff f(t) > r \geq t.$$

Hence

$$\begin{aligned}\|s\mu - \mu\|_1 &= \sum_{t\in\Gamma} |s\mu(t) - \mu(t)| \\ &= \sum_{t\in\Gamma} \int_0^1 |\chi_{F(s\mu,r)}(t) - \chi_{F(\mu,r)}(t)|dr \\ &= \int_0^1 \sum_{t\in\Gamma} |\chi_{F(s\mu,r)}(t) - \chi_{F(\mu,r)}(t)|dr \\ &= \int_0^1 |sF(\mu,r)\triangle F(\mu,r)|dr.\end{aligned}$$

Therefore

$$\varepsilon \int_0^1 |F(\mu,r)|dr = \varepsilon > \sum_{s\in E} \|s\mu - \mu\|_1 = \int_0^1 \sum_{s\in E} |sF(\mu,r)\triangle F(\mu,r)|dr.$$

Thus for some r, we must have

$$\sum_{s\in E} |sF(\mu,r)\triangle F(\mu,r)| < \varepsilon|F(\mu,r)|.$$

(3)$\Longrightarrow$(4): Take a Følner sequence (F_i), i.e., (F_i) is a sequence of finite subsets of Γ such that

$$\frac{|sF_i\triangle F_i|}{|F_i|} \to 0$$

for any $s \in \Gamma$. Set $\xi_i = |F_i|^{-1/2}\chi_{F_i} \in \ell_2\Gamma$. Observe that for finite subsets $E, F \subset \Gamma$,

$$\|\chi_E - \chi_F\|_2^2 = |E\triangle F|.$$

Hence

$$\|\lambda(s)\xi_i - \xi_i\|_2^2 = \frac{1}{|F_i|}\|\chi_{sF_i} - \chi_{F_i}\|_2^2 = \frac{|sF_i\triangle F_i|}{|F_i|} \to 0.$$

(4)$\Longrightarrow$(5): Take unit vectors $\xi_i \in \ell_2\Gamma$ with condition (4). We may assume that each ξ_i is finitely suppoted. Then $\varphi_i(s) = \langle\lambda(s)\xi_i, \xi_i\rangle$ is positive definite and $\varphi_i(s) \to \|\xi_i\|_2^2 = 1$.

(5)$\Longrightarrow$(6): We will prove it in the next section.

(6)$\Longrightarrow$(7): The trivial representation $\tau_0 \colon \Gamma \ni s \mapsto 1 \in \mathbb{C}$ extends to $C^*\Gamma = C^*_\lambda\Gamma$.

(7)$\Longrightarrow$(1): Let $\tau \colon C^*_\lambda\Gamma \to \mathbb{C}$ be any unital $*$-homomorphism, which regard it as a state. By Hahn-Banach theorem, we can extend it to $\mathbb{B}(\ell_2\Gamma)$. Since $\ell_\infty\Gamma \ni f \mapsto M_f \in \mathbb{B}(\ell_2\Gamma)$, τ is also defined on $\ell_\infty\Gamma$. Since $M_{sf} = \lambda(s)M_f\lambda(s^{-1}) \in \ell_\infty\Gamma$, we have

$$\tau(M_{sf}) = \tau(\lambda(s)M_f\lambda(s)^*) = \tau(\lambda(s))\tau(M_f)\overline{\tau(\lambda(s))} = \tau(M_f)$$

for any $s \in \Gamma$ and $f \in \ell_\infty\Gamma$, (because $\lambda(s)$ belongs to the multiplicative domain of τ). □

Remark 3.25. Let $p \geq 1$ be fixed. The condition (5) in the above can be replaced by the following:

$(5)_p$ there is a sequence (φ_i) of positive definite functions in $\ell_p\Gamma$ such that $\varphi_i(s) \to 1$ for $s \in \Gamma$,

Indeed, it is easy that $(5) \implies (5)_p$. Conversely, take $k \in \mathbb{N}$ with $k \geq p$. Then φ_i^k are positive definite such that $\varphi_i^k(s) \to 1$ and $\varphi_i^k \in \ell_1\Gamma \subset C^*_\lambda\Gamma$. Fix $i \geq 1$. Let $\|\lambda(\varphi_i^k)^{1/2}\| = c_i \geq 0$. By taking $f_i \in c_c\Gamma$ such that

$$\|\lambda(\varphi_i^k)^{1/2} - \lambda(f_i)\| < \frac{1}{2i(c_i+1)}.$$

Then we have

$$\|\lambda(\varphi_i^k) - \lambda(f_i^* * f_i)\| < \frac{1}{i}.$$

Hence for any $s \in \Gamma$,

$$|\varphi_i^k(s) - f_i^* * f_i(s)| = |\langle[\lambda(\varphi_i^k) - \lambda(f_i^* * f_i)]\delta_e, \delta_s\rangle| \leq \|\lambda(\varphi_i^k) - \lambda(f_i^* * f_i)\| \to 0.$$

It follows that $f_i^* * f_i(s) \to 1$.

4 “New” group C^*-algebras

Definition 4.1. Let π be a unitary representation of a contable discrete group Γ on a Hilbert space $\mathcal{H}$. For $\xi, \eta \in \mathcal{H}$, we denote the *matrix coefficient* of π by

$$\pi_{\xi,\eta}(s) := \langle \pi(s)\xi, \eta\rangle.$$

Note that $\pi_{\xi,\eta} \in \ell_\infty\Gamma$.

Definition 4.2. Let D be a non-zero ideal of $\ell_\infty\Gamma$. If there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\pi_{\xi,\eta} \in D$ for all $\xi, \eta \in \mathcal{H}_0$, then π is called *D-representation*. If D is invariant under the left and right translation of Γ on $\ell_\infty\Gamma$, then it is said to be *translation invariant*. In this case, D contains $c_c\Gamma$

Example 4.3. $c_c\Gamma$, $\ell_p\Gamma$, $c_0\Gamma$ are translation invariant ideals of $\ell_\infty\Gamma$.

Lemma 4.4. If π has a cyclic vector ζ such that $\pi_{\zeta,\zeta} \in D$, then π is a D-representation with respect to a dense subspace

$$\mathcal{H}_0 = \operatorname{span}\{\pi(s)\xi \colon s \in \Gamma\}.$$

Proof. Let $\xi = \pi(s)\zeta$, $\eta = \pi(t)\zeta$. Then

$$\pi_{\xi,\eta}(r) = \langle \pi(r)\xi, \eta\rangle = \langle \pi(t^{-1}rs)\zeta, \zeta\rangle = \pi_{\zeta,\zeta}(t^{-1}rs).$$

Hence $\pi_{\xi,\eta} \in D$. □

Remark 4.5. It is easy to see that λ is a c_c-representation, or a D-representation for any D.

Definition 4.6. The C^*-algebra $C^*_D\Gamma$ is the C^*-completion of the group ring $\mathbb{C}\Gamma$ by $\|\cdot\|_D$, where

$$\|f\|_D = \sup\{\|\pi(f)\| \colon \pi \text{ is a } D\text{-representation}\} \quad \text{for } f \in c_c\Gamma.$$

Remark 4.7. Note that if D_1 and D_2 are ideals of $\ell_\infty\Gamma$ with $D_1 \supset D_2$, then there exists the canonical quotient map from $C^*_{D_1}\Gamma$ onto $C^*_{D_2}\Gamma$.

Remark 4.8. Let $(\pi_i, \mathcal{H}_i)$ be a family of all D-representations of Γ with a dense subspace $\mathcal{H}_{i,0}$. Then $\pi_u = \bigoplus_i \pi_i$ is a D-representation of Γ with a dense subspace $\mathcal{H}_{u,0} = \bigoplus_{\text{finite}} \mathcal{H}_{i,0}$, which gives a faithful D-representation of $C^*_D\Gamma$. Indeed, suppose that there is $0 \neq x \in C^*_D\Gamma$ such that $\pi_u(x) = 0$. Take $f_n \in c_c\Gamma$ such that $\|f_n - x\|_D \to 0$. Then $\pi_u(f_n) \to \pi_u(x) = 0$. However $\|\pi_u(f_n)\| = \|f_n\|_D \to \|x\|_D \neq 0$, which is a contradiction.

Remark 4.9. It easily follows from the definition that $C^*_{\ell_\infty}\Gamma = C^*\Gamma$.

Lemma 4.10 (Cowling-Haagerup-Howe theorem). Let $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ be a unitary representation with a cyclic vector $\zeta \in \mathcal{H}$ such that $\pi_{\zeta,\zeta} \in \ell_2\Gamma$. Then $\|\pi(f)\| \leq \|\lambda(f)\|$ for $f \in c_c\Gamma$.

Proof. 省略. □

Theorem 4.11. $C^*_{\ell_p}\Gamma = C^*_\lambda\Gamma$ for $1 \leq p \leq 2$.

Proof. There is a canonical quotient $\Phi\colon C^*_{\ell_p}\Gamma \to C^*_\lambda\Gamma$. Suppose that $0 \neq x \in \ker\Phi$. Take a ℓ_p-representation $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ such that $\|\pi(x)\| \neq 0$. Hence there is $\zeta \in \mathcal{H}_0$ such that $\pi(x)\zeta \neq 0$. Set

$$\mathcal{H}'_0 = \text{span}\,\{\pi(s)\zeta\colon s \in \Gamma\} \subset \mathcal{H}' = \overline{\mathcal{H}'_0} \subset \mathcal{H},$$

and $\pi'(s) = \pi(s)|_{\mathcal{H}'}$ for $s \in \Gamma$. Then

$$\pi'_{\zeta,\zeta}(s) = \langle \pi(s)\zeta, \zeta\rangle \in \ell_p\Gamma,$$

and ζ is cyclic for π'. Therefore π' is ℓ_p-representation with $\pi'(x) \neq 0$. Since $\pi'_{\zeta,\zeta} \in \ell_2\Gamma$, by CHH theorem, we have $\|\pi'(f)\| \leq \|\lambda(f)\|$ for $f \in c_c\Gamma$. Take $f_n \in c_c\Gamma$ such that $\|f_n - x\|_{\ell_p} \to 0$. Then $\pi'(f_n) \to \pi'(x)$ and $\Phi(f_n) = \lambda(f_n) \to \Phi(x) = 0$, which is a contradiction. □

Lemma 4.12. Let $\varphi \in P(\Gamma)$. If $\varphi \in D$, then GNS-representation of ω_φ is D-representation.

Proof. Let ξ_φ be a corresponding cyclic vector. Then $\varphi = \pi_{\xi_\varphi,\xi_\varphi} \in D$.

□

Lemma 4.13 (Glimm's lemma). Let $A \subset \mathbb{B}(\mathcal{H})$ be a separable C^*-algebra such that $A \cap \mathbb{K}(\mathcal{H}) = \{0\}$. If $\omega \in S(A)$, then there exist orthonormal vectors (ξ_n) such that $\langle a\xi_n, \xi_n\rangle \to \varphi(a)$ for all $a \in A$.

Proof. 省略. □

Theorem 4.14. $C^*\Gamma = C^*_D\Gamma \iff$ there is positive definite $\varphi_n \in D$ such that $\varphi_n \to 1$ pointwise.

Proof. ($\Longleftarrow$) It suffices to show that the set of vector states with respect to D-representations is weak-$*$ dense in $S(C^*\Gamma)$. For $\varphi \in P(\Gamma)$, we define $\psi_n = \varphi_n\varphi \in P(\Gamma)$. Note that $\psi_n \to \varphi$ pointwise. Since $\psi_n \in D$, the GNS-representation of ψ_n is D-representation.

($\Longrightarrow$) Assume that $C^*\Gamma = C^*_D\Gamma$. Then there is a faithful D-representation of $C^*\Gamma$ with a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\pi(C^*\Gamma) \cap \mathbb{K}(\mathcal{H}) = \{0\}$. Set $A = \pi(C^*\Gamma) \subset \mathbb{B}(\mathcal{H})$. Define $\tau \in S(A)$ by $\tau(\pi(f)) = \sum_s f(s)$ for $f \in c_c\Gamma$. By Glimm's lemma, we have $\langle \pi(\delta_s)\xi_n, \xi_n\rangle \to 1$. Take $\mathcal{H}_0 \ni \xi'_n$ such that $\|\xi'_n - \xi_n\| < 1/n$. Then $\pi_{\xi'_n,\xi'_n} \in D$ is positive definite and $\pi_{\xi'_n,\xi'_n} \to 1$ pointwise. □

Corollary 4.15. (1) Γ is amenable if and only if $C^*\Gamma = C^*_{c_c}\Gamma = C^*_\lambda\Gamma$,

(2) Γ has the Haagerup property, i.e., there exists a sequence (φ_n) of positive definite functions in $c_0\Gamma$ such that $\varphi_n \to 1$ pointwise, if and only if $C^*\Gamma = C^*_{c_0}\Gamma$.

Remark 4.16. For $2 < p < \infty$, the following holds:

$$\mathrm{C}^*(\mathbb{F}_d) \overset{(1)}{=} \mathrm{C}^*_{c_0}(\mathbb{F}_d) \overset{(2)}{\neq} \mathrm{C}^*_{\ell_p}(\mathbb{F}_d) \overset{???}{\neq} \mathrm{C}^*_{\ell_2}(\mathbb{F}_d) \overset{(3)}{=} \mathrm{C}^*_\lambda(\mathbb{F}_d),$$

where

(1) by the Haagerup property,

(2) by non-amenablity,

(3) by CHH theorem.

5 Positive definite functions on $\mathbb{F}_d$

Definition 5.1. Let $\mathbb{F}_d$ be the free group on finitely many generators $a_1, \ldots, a_d$ with $d \geq 2$. We denote by $|s|$ the *word length* of $s \in \mathbb{F}_d$ with respect to the canonical generating set $\{a_1, a_1^{-1}, \ldots, a_d, a_d^{-1}\}$. For $k \geq 0$, we put

$$W_k = \{s \in \mathbb{F}_d \mid |s| = k\}.$$

We denote by χ_k the characteristic function for W_k.

Lemma 5.2. Let $q \in [1, 2]$. Let k, ℓ and m be non negative integers. Let f and g be functions on $\mathbb{F}_d$ such that $\mathrm{supp}(f) \subset W_k$ and $\mathrm{supp}(g) \subset W_\ell$, respectively. If $|k - \ell| \leq m \leq k + \ell$ and $k + \ell - m$ is even, then

$$\|(f * g)\chi_m\|_q \leq \|f\|_q \|g\|_q,$$

and if m is any other value, then

$$\|(f * g)\chi_m\|_q = 0.$$

Proof. Note that

$$(f * g)(r) = \sum_{\substack{s,t \in \mathbb{F}_d \\ r = st}} f(s)g(t) = \sum_{\substack{|s|=k \\ |t|=\ell \\ r=st}} f(s)g(t).$$

Since the possible values of $|st|$ are $|k - \ell|, |k - \ell| + 2, \ldots, k + \ell$, we have

$$\|(f * g)\chi_m\|_q = 0$$

for any other values of m.

The case where $q = 1$ is trivial. So let $q \neq 1$.

First we assume that $m = k + \ell$. If $|r| = m$, then r can be uniquely written as a product st with $|s| = k$ and $|t| = \ell$. Hence

$$(f * g)(r) = f(s)g(t).$$

Therefore

$$\|(f*g)\chi_m\|_q^q = \sum_{\substack{|st|=k+\ell\\|s|=k\\|t|=\ell}} |f(s)|^q|g(t)|^q \le \sum_{\substack{|s|=k\\|t|=\ell}} |f(s)|^q|g(t)|^q = \|f\|_q^q\|g\|_q^q.$$

Next we assume that $m = |k-\ell|, |k-\ell|+2, \ldots, k+\ell-2$. Then, we have $m = k+\ell-2j$ for $1 \le j \le \min\{k,\ell\}$. Let $r = st$ with $|r| = m$, $|s| = k$ and $|t| = \ell$. Then r can be uniquely written as a product $s't'$ such that $s = s'u$, $t = u^{-1}t'$ with $|s'| = k-j$, $|t'| = \ell-j$ and $|u| = |u^{-1}| = j$. We define

$$f'(s) = \left(\sum_{|u|=j} |f(su)|^q\right)^{\frac{1}{q}} \text{ if } |s| = k-j, \text{ and } f'(s) = 0 \text{ otherwise.}$$

We also define

$$g'(t) = \left(\sum_{|u|=j} |g(u^{-1}t)|^q\right)^{\frac{1}{q}} \text{ if } |t| = \ell-j, \text{ and } g'(t) = 0 \text{ otherwise.}$$

Note that $\mathrm{supp}(f') \subset W_{k-j}$ and $\mathrm{supp}(g') \subset W_{\ell-j}$. Moreover

$$\|f'\|_q^q = \sum_{|t|=k-j}\left(\sum_{|v|=j} |f(tv)|^q\right) = \|f\|_q^q,$$

and similarly $\|g'\|_q = \|g\|_q$. Take $2 \le p < \infty$ with $1/p + 1/q = 1$. By Hölder's inequality,

$$\begin{aligned}
|(f*g)(r)| &= \left|\sum_{\substack{|s|=k\\|t|=\ell\\r=st}} f(s)g(t)\right| = \left|\sum_{|u|=j} f(s'u)g(u^{-1}t')\right| \\
&\le \left(\sum_{|u|=j} |f(s'u)|^q\right)^{\frac{1}{q}} \left(\sum_{|u|=j} |g(u^{-1}t')|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{|u|=j} |f(s'u)|^q\right)^{\frac{1}{q}} \left(\sum_{|u|=j} |g(u^{-1}t')|^q\right)^{\frac{1}{q}} \\
&= f'(s')g'(t') = (f'*g')(r).
\end{aligned}$$

Hence $|(f*g)\chi_m| \le (f'*g')\chi_m$. Since $(k-j)+(\ell-j) = m$, it follows from the first part of the proof that

$$\|(f*g)\chi_m\|_q \le \|(f'*g')\chi_m\|_q \le \|f'\|_q\|g'\|_q = \|f\|_q\|g\|_q.$$

$\square$

Lemma 5.3. Let $1 \leq q \leq p \leq \infty$ with $1/p + 1/q = 1$. Let $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ be a unitary representation with a cyclic vector ζ such that $\pi_{\zeta,\zeta} \in \ell_p\Gamma$. Then

$$\|\pi(f)\| \leq \liminf_{n\to\infty} \left\|(f^* * f)^{(*2n)}\right\|_q^{\frac{1}{4n}}$$

for $f \in c_c\Gamma$.

Proof. For $f \in c_c\Gamma$, we set $g = f^* * f$. Then $\pi(g)$ is self-adjoint. By the spectral decomposition, for $\xi \in \mathcal{H}$ there is a regular Borel complex measure μ on $\mathbb{R}$ such that

$$\langle \pi(g)\xi, \xi\rangle = \int t d\mu(t).$$

Then

$$\begin{aligned}\|\pi(g)\xi\|^2 = \langle \pi(g)^2\xi, \xi\rangle &= \int t^2 d\mu(t)\\ &\leq \left(\int t^{2n} d\mu(t)\right)^{1/n} \left(\int 1 d\mu(t)\right)^{1-1/n}\\ &= \langle \pi(g)^{2n}\xi, \xi\rangle^{1/n} \|\xi\|^{1-1/n}\end{aligned}$$

Hence

$$\|\pi(g)\xi\| \leq \liminf_{n\to\infty} \langle \pi(g)^{2n}\xi, \xi\rangle^{1/2n}\|\xi\|.$$

If we put $\xi = \pi(h)\zeta$, $\varphi(r) = \pi_{\zeta,\zeta}(r)$ with $h \in c_c\Gamma$ and $\psi(r) = \pi_{\xi,\xi}(r)$, then

$$\psi(r) = \langle \pi(r)\pi(h)\zeta, \pi(h)\zeta\rangle = \sum_{s,t} h(s)\overline{h(t)}\varphi(t^{-1}rs).$$

Hence, $\psi \in \ell_p\Gamma$. By Hölder's inequality,

$$|\langle \pi(g)^{2n}\xi, \xi\rangle| = \left|\sum_{r\in\Gamma} g^{(*2n)}(r)\psi(r)\right| \leq \left\|g^{(*2n)}\right\|_q \|\psi\|_p.$$

Since $\mathcal{H}_0 = \{\pi(h)\zeta\colon h \in c_c\Gamma\}$ is dense in $\mathcal{H}$, we have

$$\|\pi(g)\| \leq \liminf_{n\to\infty} \left\|g^{(*2n)}\right\|_q^{\frac{1}{2n}}.$$

□

Lemma 5.4. Let k be a non negative integer. Let $1 \leq q \leq p \leq \infty$ with $1/p + 1/q = 1$. If a unitary representation π of $\mathbb{F}_d$ on a Hilbert space $\mathcal{H}$ has a cyclic vector ζ such that $\pi_{\zeta,\zeta} \in \ell_p\mathbb{F}_d$, then

$$\|\pi(f)\| \leq (k+1)\|f\|_q.$$

for $f \in c_c\mathbb{F}_d$ with $\mathrm{supp}(f) \subset W_k$.

Proof. The case where $q = 1$ and $p = \infty$ is trivial. So we may assume that $1 < q \leq 2$ and $2 \leq p < \infty$ with $1/p + 1/q = 1$.

Consider $\left\|(f^* * f)^{(*2n)}\right\|_q$. Write $f_{2j-1} = f^*$ and $f_{2j} = f$ for $j = 1, 2, \ldots, 2n$. Then

$$(f^* * f)^{(*2n)} = f_1 * f_2 * \cdots * f_{4n}.$$

We also denote $g = f_2 * \cdots * f_{4n}$. So we have

$$(f^* * f)^{(*2n)} = f_1 * g.$$

Note that $\mathrm{supp}(f_j) \subset W_k$ for $j = 1, 2, \ldots, 4n$ and $g \in c_c\mathbb{F}_d$. Put $g_\ell = g\chi_\ell$. Then $\mathrm{supp}(g_\ell) \subset W_\ell$ and

$$\|g\|_q^q = \sum_{\ell=0}^{\infty} \|g_\ell\|_q^q.$$

Here, remark that $\|g_\ell\|_q = 0$ for all but finitely many ℓ. Moreover set

$$h = f_1 * g = \sum_{\ell=0}^{\infty} f_1 * g_\ell$$

and $h_m = h\chi_m$. Then $h \in c_c\mathbb{F}_d$ and

$$\|h\|_q^q = \sum_{m=0}^{\infty} \|h_m\|_q^q.$$

Here, notice that $\|h_m\|_q = 0$ for all but finitely many m. By Lemma 5.2,

$$\|(f_1 * g_\ell)\chi_m\|_q \le \|f_1\|_q \|g_\ell\|_q$$

in the case where $|k - \ell| \le m \le k + \ell$ and $k + \ell - m$ is even. Hence

$$\|h_m\|_q = \left\| \sum_{\ell=0}^{\infty} (f_1 * g_\ell)\chi_m \right\|_q \le \sum_{\ell=0}^{\infty} \|(f_1 * g_\ell)\chi_m\|_q \le \|f_1\|_q \sum_{\substack{\ell=|m-k| \\ m+k-\ell \text{ even}}}^{m+k} \|g_\ell\|_q.$$

By writing $\ell = m + k - 2j$,

$$\begin{aligned}
\|h_m\|_q &\le \|f_1\|_q \sum_{j=0}^{\min\{m,k\}} \|g_{m+k-2j}\|_q \\
&\le \|f_1\|_q \left(\sum_{j=0}^{\min\{m,k\}} \|g_{m+k-2j}\|_q^q \right)^{\frac{1}{q}} \left(\sum_{j=0}^{\min\{m,k\}} 1^p \right)^{\frac{1}{p}} \\
&\le (k+1)^{\frac{1}{p}} \|f_1\|_q \left(\sum_{j=0}^{\min\{m,k\}} \|g_{m+k-2j}\|_q^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Then

$$\begin{aligned}
\|h\|_q^q = \sum_{m=0}^{\infty} \|h_m\|_q^q &\le (k+1)^{\frac{q}{p}} \|f_1\|_q^q \sum_{m=0}^{\infty} \sum_{j=0}^{\min\{m,k\}} \|g_{m+k-2j}\|_q^q \\
&= (k+1)^{\frac{q}{p}} \|f_1\|_q^q \sum_{j=0}^{k} \sum_{m=j}^{\infty} \|g_{m+k-2j}\|_q^q \\
&= (k+1)^{\frac{q}{p}} \|f_1\|_q^q \sum_{j=0}^{k} \sum_{\ell=k-j}^{\infty} \|g_\ell\|_q^q \\
&\le (k+1)^{\frac{q}{p}} \|f_1\|_q^q \sum_{j=0}^{k} \|g\|_q^q \\
&= (k+1)^{\frac{q}{p}+1} \|f_1\|_q^q \|g\|_q^q.
\end{aligned}$$

Hence $\|f_1 * g\|_q \le (k+1)\|f_1\|_q\|g\|_q$. Therefore we inductively get,

$$\|f_1 * (f_2 * \cdots * f_{4n})\|_q \le (k+1)\|f_1\|_q\|f_2 * \cdots * f_{4n}\|_q \le \cdots \le (k+1)^{4n-1}\|f\|_q^{4n}.$$

Thus it follows from Lemma 5.3 that

$$\|\pi(f)\| \le \liminf_{n\to\infty} \left\|(f^* * f)^{(*2n)}\right\|_q^{\frac{1}{4n}} \le (k+1)\|f\|_q.$$

□

Remark 5.5. For $0 < \alpha < 1$, we set $\varphi_\alpha(s) = \alpha^{|s|}$, and it is positive definite on $\mathbb{F}_d$ by [Ha, Lemma 1.2].

Theorem 5.6. Let $2 \le p < \infty$. Let φ be a positive definite function on $\mathbb{F}_d$. Then the following conditions are equivalent:

(1) φ can be extended to the positive linear functional on $C^*_{\ell_p}\mathbb{F}_d$.

(2) $\sup_k |\varphi\chi_k|_p(k+1)^{-1} < \infty$.

(3) The function $s \mapsto \varphi(s)(1+|s|)^{-1-\frac{2}{p}}$ belongs to $\ell_p\mathbb{F}_d$.

(4) For any $\alpha \in (0,1)$, the function $s \mapsto \varphi(s)\alpha^{|s|}$ belongs to $\ell_p\mathbb{F}_d$.

Proof. We may assume that $\varphi(e) = 1$.

(1)$\Longrightarrow$(2): It follows from (1) that ω_φ extends to the state on $C^*_{\ell_p}\mathbb{F}_d$. Hence for $f \in c_c\mathbb{F}_d$, we have

$$|\omega_\varphi(f)| \le \|f\|_{\ell_p}.$$

If we put $f = |\varphi|^{p-2}\overline{\varphi}\chi_k$, then

$$|\omega_\varphi(f)| = |\varphi\chi_k|_p^p.$$

Let π be an ℓ_p-representation of $\mathbb{F}_d$ on a Hilbert space $\mathcal{H}$ with a dense subspace $\mathcal{H}_0$. Then

$$\|\pi(f)\|^2 = \sup_{\substack{\xi\in\mathcal{H}_0\\ \|\xi\|=1}} \langle \pi(f^* * f)\xi, \xi\rangle_{\mathcal{H}}.$$

Fix $\zeta \in \mathcal{H}_0$ with $\|\zeta\| = 1$. We denote by σ the restriction of π onto the subspace

$$\mathcal{H}_\sigma = \overline{\operatorname{span}}\{\pi(s)\zeta \colon s \in \mathbb{F}_d\} \subset \mathcal{H}.$$

Then

$$\langle \pi(f^* * f)\xi, \xi\rangle_{\mathcal{H}} = \langle \sigma(f^* * f)\xi, \xi\rangle_{\mathcal{H}_\sigma}.$$

Since ζ is cyclic for σ such that $\sigma_{\xi,\xi} \in \ell_p(\mathbb{F}_d)$, by Lemma 5.4,

$$\|\sigma(f)\| \le (k+1)\|f\|_q.$$

Hence

$$\|\sigma(f^* * f)\| = \|\sigma(f)\|^2 \le (k+1)^2\|f\|_q^2.$$

Therefore we obtain

$$\|f\|_{\ell_p}^2 = \sup\{\|\pi(f)\|^2 \colon \pi \text{ is an } \ell_p\text{-representation}\} \le (k+1)^2\|f\|_q^2 = (k+1)^2\|\varphi\chi_k\|_p^{2(p-1)},$$

namely,

$$\|f\|_{\ell_p} \le (k+1)\|\varphi\chi_k\|_p^{p-1}.$$

Consequently,

$$\|\varphi\chi_n\|_p \leq k+1.$$

(2)$\Longrightarrow$(3)$\Longrightarrow$(4): Easy.

(4)$\Longrightarrow$(1): Note that $\psi_\alpha = \varphi\varphi_\alpha$ is also positive definite. By the GNS construction, we obtain the unitary representation π_α of $\mathbb{F}_d$ with the cyclic vector ξ_α such that for $f \in c_c\mathbb{F}_d$,

$$\omega_{\psi_\alpha}(f) = \langle \pi_\alpha(f)\xi_\alpha, \xi_\alpha\rangle.$$

Since π_α is an ℓ_p-representation, ω_{ψ_α} can be seen as a state on $C^*_{\ell_p}\mathbb{F}_d$. By taking the weak-$*$ limit of ω_{ψ_α} as $\alpha \nearrow 1$, we conclude that ω_φ can be extended to the state on $C^*_{\ell_p}\mathbb{F}_d$. □

Corollary 5.7. Let $p \in [2,\infty)$ and $\alpha \in (0,1)$. The positive definite function φ_α can be extended to the state on $C^*_{\ell_p}\mathbb{F}_d$ if and only if

$$\alpha \leq (2d-1)^{-\frac{1}{p}}.$$

Proof. It follows from the fact $\varphi_\alpha \in \ell_p\mathbb{F}_d \iff \alpha < (2d-1)^{-\frac{1}{p}}$. **[Problem 14]** □

Corollary 5.8. For $2 \leq q < p \leq \infty$, the canonical quotient map from $C^*_{\ell_p}\mathbb{F}_d$ onto $C^*_{\ell_q}\mathbb{F}_d$ is not injective.

Proof. It suffices to consider the case where $p \neq \infty$, because $\mathbb{F}_d$ is not amenable.

Suppose that the canonical quotient map from $C^*_{\ell_p}\mathbb{F}_d$ onto $C^*_{\ell_q}\mathbb{F}_d$ is injective for some $q < p$. Take a real number α with

$$(2d-1)^{-\frac{1}{q}} < \alpha \leq (2d-1)^{-\frac{1}{p}}.$$

By using Corollary 5.7,

$$|\omega_{\varphi_\alpha}(f)| \leq \|f\|_{\ell_p} = \|f\|_{\ell_q} \quad \text{for } f \in c_c\mathbb{F}_d.$$

Therefore it follows that ω_{φ_α} can be also extended to the state on $C^*_{\ell_q}\mathbb{F}_d$, but it contradicts to the choice of α. □

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