BOUNDS FOR AN OPERATOR CONCAVE FUNCTION

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Abstract. We shall provide new bounds for the difference of the Davis-Choi-Jensen inequality. Among others, we show that if Φ is a unital positive linear map and f is operator convcave on an interval $[m, M]$, then

$$f(\Phi(A)) - \Phi(f(A)) \leq 2\left(f\left(\frac{m+M}{2}\right) - \frac{f(m)+f(M)}{2}\right) I$$

for every selfadjoint operator A such that $mI \leq A \leq MI$ for some scalars $m < M$. Moreover, we discuss an external version of the Davis-Choi-Jensen inequality.

Key words. Operator concave function, Davis-Choi-Jensen inequality, Positive linear map

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1. Introduction. Let Φ be a unital positive linear map from $\mathcal{B}(H)$ to $\mathcal{B}(K)$, where $\mathcal{B}(H)$ is the C*-algebra of all bounded linear operators on a Hilbert space H. The Davis-Choi-Jensen inequality [1, 2] says that if a real-valued function f is operator concave on an interval J, then

$$\Phi(f(A)) \leq f(\Phi(A)) \tag{1.1}$$

for every selfadjoint operator A with spectrum $\sigma(A) \subset J$. Though in the case of concave function the inequality (1.1) does not hold in general, we have the following estimate [7]: If f is concave and A is a selfadjoint operator on H such that $mI \leq A \leq MI$ for some scalars $m < M$, then

$$-\mu(m, M, f)I \leq f(\Phi(A)) - \Phi(f(A)) \leq \mu(m, M, f)I \tag{1.2}$$

for all unital positive linear maps Φ where the bound $\mu(m, M, f)$ of f is defined by

$$\mu(m, M, f) = \max\left\{f(t) - \frac{f(M)-f(m)}{M-m}(t-m) - f(m) : t \in [m, M]\right\}. \tag{1.3}$$

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We note that bounds of (1.2) are sharp.

In [3], the first author showed the following estimate for the normalized Jensen functional: If a real-valued function f is concave on a convex set C, then for each positive n-tuples $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ with $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n q_i = 1$

$$\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\left(f\left(\sum_{i=1}^n q_i x_i\right) - \sum_{i=1}^n q_i f(x_i)\right) \le \left(f\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i f(x_i)\right)$$
$$\le \max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\left(f\left(\sum_{i=1}^n q_i x_i\right) - \sum_{i=1}^n q_i f(x_i)\right)$$

for all $(x_1, \cdots, x_n) \in C^n$.

In this note, based on the idea of [3], we shall provide new bounds for the difference of the Davis-Choi-Jensen inequality. Among others, we show that if Φ is a unital positive linear map and f is operator concave on an interval $[m, M]$, then

$$f(\Phi(A)) - \Phi(f(A)) \le 2\left(f\left(\frac{m+M}{2}\right) - \frac{f(m)+f(M)}{2}\right) I$$

for every selfadjoint operator A such that $mI \le A \le MI$ for some scalars $m < M$. Moreover, we discuss an external version of the Davis-Choi-Jensen inequality.

2. Davis-Choi-Jensen inequality. Let Φ and Ψ be two positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. Φ is said to be α-upper dominant by Ψ if there exists $\alpha > 0$ such that $\alpha\Psi \ge \Phi$. Similarly Φ is said to be β-lower dominant by Ψ if there exists $\beta > 0$ such that $\Phi \ge \beta\Psi$. Moreover, Φ is (α, β)-dominant by Ψ if Φ is α-upper and β-lower dominant by Ψ. The vector $(p_1, \cdots, p_n)$ is said to be a weight vector if $p_i > 0$ for all $i = 1, \cdots, n$ and $\sum_{i=1}^n p_i = 1$. For example, we put two positive linear maps Φ and $\Psi : \mathcal{B}(H) \oplus \cdots \oplus \mathcal{B}(H) \mapsto \mathcal{B}(H)$ as follows:

$$\Phi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^n p_i A_i \qquad \text{and} \qquad \Psi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^n q_i A_i,$$

where $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ are weight vectors. If we put $\alpha = \max_{1\le i\le n}\{\frac{p_i}{q_i}\}$ and $\beta = \min_{1\le i\le n}\{\frac{p_i}{q_i}\}$, then it follows that Φ is (α, β)-dominant by Ψ. In fact, we have

$$(\alpha\Psi - \Phi)(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^n (\alpha - \frac{p_i}{q_i}) q_i A_i$$

and

$$(\Phi - \beta\Psi)(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^n (\frac{p_i}{q_i} - \beta) q_i A_i.$$

Therefore $\alpha\Psi - \Phi$ and $\Phi - \beta\Psi$ are positive linear maps.

Firstly, we provide bounds for the difference of the Davis-Choi-Jensen inequality:

THEOREM 2.1. *Let Φ and Ψ be two unital positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that Φ is (α,β)-dominant by Ψ. If f is operator concave on an interval J, then*

$$\beta\left(f(\Psi(A)) - \Psi(f(A))\right) \le f(\Phi(A)) - \Phi(f(A)) \le \alpha\left(f(\Psi(A)) - \Psi(f(A))\right) \tag{2.1}$$

for every selfadjoint operator A with spectrum $\sigma(A) \subset J$.

Proof. If we put $\Phi_0(X) = \frac{1}{\alpha}X$, then Φ_0 is a positive linear map. Since Φ is α-upper dominant by Ψ, we have $\Psi - \frac{1}{\alpha}\Phi$ is positive and $(\Psi - \frac{1}{\alpha}\Phi)(I) + \Phi_0(I) = I$. Therefore, by the Jensen operator inequality (1.1) we have

$$\begin{aligned} f(\Psi(A)) &= f(\Psi(A) - \frac{1}{\alpha}\Phi(A) + \frac{1}{\alpha}\Phi(A)) = f\left((\Psi - \frac{1}{\alpha}\Phi)(A) + \Phi_0(\Phi(A))\right) \\ &\ge (\Psi - \frac{1}{\alpha}\Phi)(f(A)) + \Phi_0(f(\Phi(A))) \\ &= \Psi(f(A)) - \frac{1}{\alpha}\Phi(f(A)) + \frac{1}{\alpha}f(\Phi(A)) \end{aligned}$$

and this implies the second inequality of (2.1).

Similarly, if we put $\Phi_1(X) = \beta X$, then it follows that

$$\begin{aligned} f(\Phi(A)) &= f\left(\Phi(A) - \beta\Psi(A) + \beta\Psi(A)\right) = f\left((\Phi - \beta\Psi)(A) + \Phi_1(\Psi(A))\right) \\ &\ge (\Phi - \beta\Psi)(f(A)) + \Phi_1(f(\Psi(A))) = \Phi(f(A)) - \beta\Psi(f(A)) + \beta f(\Psi(A)). \end{aligned}$$

□

By Theorem 2.1, we have the following corollary as an operator concave version of [3, Theorem 1], see also [4].

COROLLARY 2.2. *Let $(p_1,\cdots,p_n)$ and $(q_1,\cdots,q_n)$ be two weight vectors. If f is operator concave on an interval J, then*

$$\begin{aligned} &\beta\left(f\left(\sum_{i=1}^n q_iA_i\right) - \sum_{i=1}^n q_if(A_i)\right) \\ &\le f\left(\sum_{i=1}^n p_iA_i\right) - \sum_{i=1}^n p_if(A_i) \le \alpha\left(f\left(\sum_{i=1}^n q_iA_i\right) - \sum_{i=1}^n q_if(A_i)\right) \end{aligned}$$

for all selfadjoint operators $A_1,\cdots,A_n$ such that $\sigma(A_i) \subset J$ for all $i = 1,\cdots,n$, where $\alpha = \max_{1\le i\le n}\{\frac{p_i}{q_i}\}$ and $\beta = \min_{1\le i\le n}\{\frac{p_i}{q_i}\}$.

In particular,

$$n \min_{1\le i\le n}\{p_i\}\left(f\left(\sum_{i=1}^{n}\frac{1}{n}A_i\right)-\sum_{i=1}^{n}\frac{1}{n}f(A_i)\right)$$
$$\le f\left(\sum_{i=1}^{n}p_iA_i\right)-\sum_{i=1}^{n}p_if(A_i)\le n\max_{1\le i\le n}\{p_i\}\left(f\left(\sum_{i=1}^{n}\frac{1}{n}A_i\right)-\sum_{i=1}^{n}\frac{1}{n}f(A_i)\right).$$

The following corollary is a two variable version of Theorem 2.1.

COROLLARY 2.3. *Let Φ, Ψ, Φ' and Ψ' be positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that $\Phi(I)+\Psi(I)=I$ and $\Phi'(I)+\Psi'(I)=I$ and Φ is (α,β)-dominant by Φ' and Ψ is (α,β)-dominant by Ψ'. If a real-valued function f is operator concave on an interval J, then*

$$\begin{aligned}&\beta\left(f(\Phi'(A)+\Psi'(B))-(\Phi'(f(A))+\Psi'(f(B))\right)\\ &\le f(\Phi(A)+\Psi(B))-(\Phi(f(A))+\Psi(f(B)))\\ &\le \alpha\left(f(\Phi'(A)+\Psi'(B))-(\Phi'(f(A))+\Psi'(f(B))\right)\end{aligned}$$

for all selfadjoint operators A and B with $\sigma(A)$, $\sigma(B)$, $\sigma(\Phi(A)+\Psi(B))$ and $\sigma(\Phi'(A)+\Psi'(B))\subset J$.

REMARK 2.4. *Similarly we have the following n-variable version of Corollary 3. Let $\{\Phi_i\}$ and $\{\Phi_i'\}$ be positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that $\sum_{i=1}^n\Phi_i(I)=\sum_{i=1}^n\Phi_i'(I)=I$ and Φ_i is (α,β)-dominant by Φ_i' for $i=1,\cdots,n$. If a real-valued function f is operator concave on an interval J, then*

$$\beta\left(f\left(\sum_{i=1}^{n}\Phi_i'(A_i)\right)-\sum_{i=1}^{n}\Phi_i'(f(A_i))\right)\le f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right)-\sum_{i=1}^{n}\Phi_i(f(A_i))$$
$$\le\alpha\left(f\left(\sum_{i=1}^{n}\Phi_i'(A_i)\right)-\sum_{i=1}^{n}\Phi_i'(f(A_i))\right)$$

for all selfadjoint operators A and B with $\sigma(A)$, $\sigma(B)$, $\sigma(\sum_{i=1}^n\Phi_i(A))$ and $\sigma(\sum_{i=1}^n\Phi_i'(A))\subset J$.

In the case of a concave function, we have no relation between $\Phi(f(A))$ and $f(\Phi(A))$. Though we have the estimate of (1.2), we provide new bounds for the difference of the Davis-Choi-Jensen inequality by means of the difference of concavity.

THEOREM 2.5. *Let Φ be a unital positive linear map from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. If a real-valued function $f(t)$ is concave on $[m,M]$, then*

$$-\frac{2}{M-m}\left(f\left(\frac{m+M}{2}\right)-\frac{f(m)+f(M)}{2}\right)\Phi(F(A))$$

$$\leq f(\Phi(A)) - \Phi(f(A)) \leq \frac{2}{M-m}\left(f\left(\frac{m+M}{2}\right) - \frac{f(m)+f(M)}{2}\right) F(\Phi(A))$$

for every selfadjoint operator A such that $mI \leq A \leq MI$ for some scalars $m < M$, where a real-valued function $F(t)$ on $[m, M]$ is defined by

$$F(t) = \frac{M-m}{2} + \left| t - \frac{M+m}{2} \right|.$$

Proof. Since Φ is a unital positive linear map and f is concave on $[m, M]$, we have

$$\begin{aligned}\Phi(f(A)) &\geq \Phi\left(\frac{f(M)-f(m)}{M-m}A + \frac{Mf(m)-mf(M)}{M-m}I\right) \\ &= \frac{f(M)-f(m)}{M-m}\Phi(A) + \frac{Mf(m)-mf(M)}{M-m}I.\end{aligned}$$

On the other hand, it follows from (1.4) that

$$\begin{aligned}&f(t) - \frac{f(M)-f(m)}{M-m}t + \frac{Mf(m)-mf(M)}{M-m} \\ &= f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) - \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \\ &\leq \frac{2}{M-m}\max\{M-t, t-m\}\left(f\left(\frac{m+M}{2}\right) - \frac{f(m)+f(M)}{2}\right) \\ &= \frac{2}{M-m}\left(\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right)F(t)\end{aligned}$$

and this implies

$$f(\Phi(A)) - \Phi(f(A)) \leq \frac{2}{M-m}\left(\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right)F(\Phi(A)).$$

For the first half of Theorem 2.5, we have

$$\begin{aligned}f(\Phi(A)) - \Phi(f(A)) &\geq \frac{f(M)-f(m)}{M-m}\Phi(A) + \frac{Mf(m)-mf(M)}{M-m}I - \Phi(f(A)) \\ &= \Phi(\frac{f(M)-f(m)}{M-m}A + \frac{Mf(m)-mf(M)}{M-m}I - f(A)) \\ &\geq -\frac{2}{M-m}\left(\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right)\Phi(F(A)).\end{aligned}$$

□

Since $\frac{1}{M-m}F(\Phi(A)) \leq I$ in Theorem 2.5, we have the following upper bound for the difference of the Davis-Choi-Jensen inequality:

Corollary 2.6. *Let Φ, f and A be as in Theorem 2.5. Then*

$$f(\Phi(A))-\Phi(f(A))\le 2\left(f\left(\frac{m+M}{2}\right)-\frac{f(m)+f(M)}{2}\right)I.$$

The following corollary is another expression of (1.2).

Corollary 2.7. *Let Φ, f and A be as in Theorem 2.5. Then*

$$-\left(\tilde{f}_{\max}-\frac{f(m)+f(M)}{2}\right)I\le f(\Phi(A))-\Phi(f(A))\le\left(\tilde{f}_{\max}-\frac{f(m)+f(M)}{2}\right)I,$$

where $\tilde{f}(t)=f(t)-\frac{f(M)-f(m)}{M-m}t+\frac{(M+m)(f(M)-f(m))}{2(M-m)}$ *and* $\tilde{f}_{\max}=\max\{\tilde{f}(t): m\le t\le M\}$.

Proof. Since

$$\begin{aligned}f(t)-\frac{f(M)-f(m)}{M-m}t-\frac{Mf(m)-mf(M)}{M-m}&=\tilde{f}(t)-\frac{f(m)+f(M)}{2}\\&\le\tilde{f}_{\max}-\frac{f(m)+f(M)}{2},\end{aligned}$$

it follows from the concavity of f that

$$\begin{aligned}f(\Phi(A))-\Phi(f(A))&\le f(\Phi(A))-\frac{f(M)-f(m)}{M-m}\Phi(A)-\frac{Mf(m)-mf(M)}{M-m}I\\&\le\left(\tilde{f}_{\max}-\frac{f(m)+f(M)}{2}\right)I.\end{aligned}$$

On the other hand, by Stinespring decomposition theorem [10], Φ restricted to a C^*-algebra $C^*(A)$ generated by A and I admits a decomposition $\Phi(X)=C^*\phi(X)C$ for all $X\in C^*(A)$, where ϕ is a $*$-representation of $C^*(A)\subset B(H)$ and C is a bounded linear operator from K to a Hilbert space K'. Since Φ is unital, we have $C^*C=I$. For every unit vector $x\in K$,

$$\begin{aligned}&\langle f(\Phi(A))x,x\rangle-\langle\Phi(f(A))x,x\rangle=\langle f(C^*\phi(A)C)x,x\rangle-\langle C^*\phi(f(A))Cx,x\rangle\\&=\langle f(\phi(A))Cx,Cx\rangle-\langle f(C^*\phi(A)C)x,x\rangle\\&\le f\left(\langle\phi(A)Cx,Cx\rangle\right)-\langle f(C^*\phi(A)C)x,x\rangle\\&\le f(\langle C^*\phi(A)Cx,x\rangle)-\frac{f(M)-f(m)}{M-m}\langle C^*\phi(A)Cx,x\rangle-\frac{Mf(m)-mf(M)}{M-m}\\&\le\tilde{f}_{\max}-\frac{f(m)+f(M)}{2}.\end{aligned}$$

Therefore, we have

$$\Phi(f(A))-f(\Phi(A))\le\left(\tilde{f}_{\max}-\frac{f(m)+f(M)}{2}\right)I$$

and this implies the first half part of the desired inequality. □

3. External version of Davis-Choi-Jensen inequality. In this section, we consider bounds of operator concavity in terms of an external formula. A real-valued continuous function f on J is operator concave if and only if

$$f((1+p)A - pB) \leq (1+p)f(A) - pf(B) \tag{3.1}$$

for all $p > 0$ and all selfadjoint operators A and B with $\sigma(A), \sigma(B)$ and $\sigma((1+p)A - pB) \subset J$. Then we have the following external version of the Jensen operator inequality: If f is operator concave, then

$$f\left((1+\sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_i B_i\right) \leq (1+\sum_{i=1}^{n} p_i)f(A) - \sum_{i=1}^{n} p_i f(B_i) \tag{3.2}$$

for all selfadjoint operators A and B_i $(i = 1, \cdots, n)$ with $\sigma(A), \sigma(B_i)$ and $\sigma((1+\sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_i B_i) \subset J$, also see [5, 9].

For a real-valued continuous function f, we define the following notation

$$A \ \sigma_f \ B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

for positive invertible A and selfadjoint B, also see [8].

Let Φ be a positive linear map from $\mathcal{B}(H)$ to $\mathcal{B}(K)$. In [6], we show the following external version of the Davis-Choi-Jensen inequality: Let f be a real-valued continuous function on an interval J. Then f is operator convcave if and only if

$$f(\Phi(A) - \Psi(B)) \leq \Phi(I) \ \sigma_f \ \Phi(A) - \Psi(f(B)) \tag{3.3}$$

for all positive linear maps Φ, Ψ such that $\Phi(I) - \Psi(I) = I$ and for all selfadjoint operators A and B with $\sigma(A), \sigma(B)$ and $\sigma(\Phi(A) - \Psi(B)) \subset J$. The invertibility of $\Phi(I)$ guarantees the formulation of (3.3). In this case, we have

$$\Phi(f(A)) \leq \Phi(I) \ \sigma_f \ \Phi(A).$$

In fact, in Stinespring decomposition theorem $\Phi(X) = C^*\phi(X)C$, we have the polar decomposition $C = V|C|$ such that $|C|$ is invertible, because $C^*C = \Phi(I) = I + \Psi(I) > 0$. Since $V^*V = I$, it follows that

$$\begin{aligned}\Phi(f(A)) &= |C|V^* f(\phi(A))V|C| \leq |C| f(V^*\phi(A)V)|C| \\ &= |C| f(|C|^{-1} C^* \phi(A) C |C|^{-1})|C| = \Phi(I) \ \sigma_f \ \Phi(A).\end{aligned}$$

If moreover C is invertible, then we have $\Phi(f(A)) = \Phi(I) \ \sigma_f \ \Phi(A)$ and hence

$$f(\Phi(A) - \Psi(B)) \leq \Phi(f(A)) - \Psi(f(B)), \tag{3.4}$$

see [6].

Based on the external version (3.4) of the Davis-Choi-Jensen inequality, we have the following bounds for the difference of the operator concavity.

THEOREM 3.1. *Let Φ, Ψ, Φ' and Ψ' be positive linear maps from $\mathcal{B}(H)$ to $\mathcal{B}(K)$ such that $\Phi(I)-\Psi(I)=I$ and $\Phi'(I)-\Psi'(I)=I$ and Φ is (β,α)-dominant by Φ' and Ψ is (α,β)-dominant by Ψ'. If a real-valued function f is operator concave on an interval J, then*

$$\begin{aligned}&\beta\left(\Phi'(f(A))-\Psi'(f(B))-f(\Phi'(A)-\Psi'(B))\right)\\ \le\ &\Phi(f(A))-\Psi(f(B))-f(\Phi(A)-\Psi(B))\\ \le\ &\alpha\left(\Phi'(f(A))-\Psi'(f(B))-f(\Phi'(A)-\Psi'(B))\right)\end{aligned}$$

for all selfadjoint operators A and B with $\sigma(A)$, $\sigma(B)$, $\sigma(\Phi(A)-\Psi(B))$ and $\sigma(\Phi'(A)-\Psi'(B))\subset J$.

Proof. Put $\Phi_1(X)=\alpha X$. Since Φ is α-lower dominant by Φ' and Ψ is α-upper dominant by Ψ', and $(\Phi-\alpha\Phi')(I)+(\alpha\Psi'-\Psi)(I)+\Phi_1(I)=I$, it follows from the operator concavity of f that

$$\begin{aligned}&f(\Phi(A)-\Psi(B))=f((\Phi-\alpha\Phi')(A)+(\alpha\Psi'-\Psi)(B)+\Phi_1(\Phi'(A)-\Psi'(B)))\\ &\ge(\Phi-\alpha\Phi')(f(A))+(\alpha\Psi'-\Psi)(f(B))+\Phi_1(f(\Phi'(A)-\Psi'(B)))\\ &=\Phi(f(A))-\Psi(f(B))-\alpha\left(f(\Phi'(A)-\Psi'(B))-(\Phi'(f(A))-\Psi'(f(B)))\right)\end{aligned}$$

for all selfadjoint operators A and B with $\sigma(A)$, $\sigma(B)$, $\sigma(\Phi(A)-\Psi(B))$ and $\sigma(\Phi'(A)-\Psi'(B))\subset J$. This fact implies the second half of Theorem 3.1. Similarly we have the first half of Theorem 3.1. □

Finally we show an application of Theorem 3.1. Put positive linear maps Φ, Ψ, Φ' and Ψ' : $\mathcal{B}(H)\oplus\cdots\oplus\mathcal{B}(H)\mapsto\mathcal{B}(H)$ as follows:

$$\Phi(A_1\oplus\cdots\oplus A_n)=\sum_{i=1}^n\frac{1+\sum_{i=1}^n p_i}{n}A_i\qquad\text{and}\qquad\Psi(A_1\oplus\cdots\oplus A_n)=\sum_{i=1}^n p_iA_i.$$

$$\Phi'(A_1\oplus\cdots\oplus A_n)=\sum_{i=1}^n\frac{1+\sum_{i=1}^n q_i}{n}A_i\qquad\text{and}\qquad\Psi'(A_1\oplus\cdots\oplus A_n)=\sum_{i=1}^n q_iA_i,$$

where $p_i,q_i>0$ for $i=1,\cdots,n$. Then it follows that $\Phi(I)-\Psi(I)=I$ and $\Phi'(I)-\Psi'(I)=I$. If we put $\alpha=\max\{\frac{p_i}{q_i}\}$ and $\frac{1+\sum_{i=1}^n p_i}{1+\sum_{i=1}^n q_i}>\alpha$, then Φ is α-lower dominant of Φ' and Ψ is α-upper dominant of Ψ'. If we put $\beta=\min\{\frac{p_i}{q_i}\}$ and $\frac{1+\sum_{i=1}^n p_i}{1+\sum_{i=1}^n q_i}<\beta$, then Φ is β-upper dominant of Φ' and Ψ is β-lower dominant of Ψ'. Hence by Theorem 3.1, we obtain the following external version of Corollary 2.2:

COROLLARY 3.2. *Let f be operator convex on an interval J and A and B_i $(i=1,\cdots,n)$ selfadjoint operators with $\sigma(A),\sigma(B_i)$ and $\sigma((1+\sum_{i=1}^n p_i)A-\sum_{i=1}^n p_iB_i)\subset$*

J. Let $\alpha = \max\{\frac{p_i}{q_i}\}$ *and* $\beta = \min\{\frac{p_i}{q_i}\}$. *If* $\beta > \frac{1+\sum p_i}{1+\sum q_i}$, *then*

$$\beta\left(f\left((1+\sum_{i=1}^{n} q_i)A - \sum_{i=1}^{n} q_i B_i\right) - ((1+\sum_{i=1}^{n} q_i)f(A) - \sum_{i=1}^{n} q_i f(B_i))\right)$$
$$\leq f\left((1+\sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_i B_i\right) - ((1+\sum_{i=1}^{n} p_i)f(A) - \sum_{i=1}^{n} p_i f(B_i))$$

and if $\frac{1+\sum p_i}{1+\sum q_i} > \alpha$, *then*

$$f\left((1+\sum_{i=1}^{n} p_i)A - \sum_{i=1}^{n} p_i B_i\right) - ((1+\sum_{i=1}^{n} p_i)f(A) - \sum_{i=1}^{n} p_i f(B_i))$$
$$\leq \alpha\left(f\left((1+\sum_{i=1}^{n} q_i)A - \sum_{i=1}^{n} q_i B_i\right) - ((1+\sum_{i=1}^{n} q_i)f(A) - \sum_{i=1}^{n} q_i f(B_i))\right).$$

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