ON THE ANDO-LI-MATHIAS MEAN AND THE KARCHER MEAN OF POSITIVE DEFINITE MATRICES

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Abstract. In this paper, from the viewpoint of the Ando-Hiai inequality, we make a comparison between the Ando-Li-Mathias geometric mean and the Karcher mean of n positive definite matrices. Among others, we show complements of the n-variable Ando-Hiai inequality for the Ando-Li-Mathias geometric mean by means of the Kantorovich constant.

1. Introduction

Let $\mathbb{M} = \mathbb{M}_d$ be the set of all $d \times d$ matrices on the complex number field $\mathbb{C}$, $\mathbb{P} = \mathbb{P}_d$ be the set of all $d \times d$ positive definite matrices and I stands for the identity matrix. For Hermitian matrices A, B we write $A \geq B$ or $B \leq A$ to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ indicates that A is positive semidefinite. This is known as the Löwner partial order, or the usual order. If A is positive definite, that is, positive semidefinite and invertible, we write $A > 0$. For two positive semidefinite matrices A and B, the matrix geometric mean $A \sharp_\alpha B$ is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}} \qquad \text{for all } 0 \leq \alpha \leq 1$$

if $A > 0$. In the case of $\alpha = \frac{1}{2}$, we denote $A \sharp_{1/2} B$ by $A \sharp B$ simply. In 2004, Ando, Li and Mathias [2] succeeded in the formulation of the geometric mean for n positive definite matrices, and they showed that it has many required properties as the geometric mean. The weighted version of the Ando-Li-Mathias geometric mean was established by Lawson and Lim [19]. Following [2], we recall the definition of the *Ando-Li-Mathias geometric mean* $G_{\rm alm}(A_1, \cdots, A_n)$ for n positive definite matrices $A_1, \cdots, A_n$. We simply call it the *ALM mean.* Let $G_{\rm alm}(A_1, A_2) = A_1 \sharp A_2$. For $n \geq 3$, $G_{\rm alm}(A_1, \cdots, A_n)$ is definied inductively as follows: Put $A_i^{(0)} = A_i$ for all $i = 1, \ldots, n$ and

$$A_i^{(r)} = G_{\rm alm}((A_j^{(r-1)})_{j \neq i}) = G_{\rm alm}(A_1^{(r-1)}, \cdots, A_{i-1}^{(r-1)}, A_{i+1}^{(r-1)}, \cdots, A_n^{(r-1)})$$

inductively for r. Then the sequences $\{A_i^{(r)}\}$ have the same limit for all $i = 1, \ldots, n$ in the Thompson metric $d(A, B) = \| \log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \|_\infty$ for positive definite A and B, where the spectral (operator) norm of $X \in \mathbb{M}_d$ is defined by $\| X \|_\infty \equiv \max\{ \| Xx \| : \| x \| = 1, x \in$

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$\mathbb{C}^d\}$. So we can define $G_{\rm alm}(A_1,\cdots,A_n)=\lim_{r\to\infty}A_i^{(r)}$. Then the arithmetic-geometric-harmonic mean inequality holds:

$$\left(\frac{1}{n}\sum_{i=1}^n A_i^{-1}\right)^{-1}\le G_{\rm alm}(A_1,\cdots,A_n)\le\frac{1}{n}\sum_{i=1}^n A_i.$$

Since then, many authors have studied geometric means of n-matrices [10, 17, 18]. On the other hand, Moakher [21] and then Bhatia and Holbrook [8] suggested a new definition of the geometric mean for n positive definite matrices by taking the mean to be the unique minimizer of the sum of the squares of the distances $\delta_2(A,B)=\|\log A^{-\frac12}BA^{-\frac12}\|_2$ with the Hilbert-Schmidt norm $\|A\|_2=\sqrt{{\rm tr}(A^*A)}$. Computing appropriate derivatives as in [21, 6] yields that it coincides with the unique positive definite solution of the *Karcher equation*

$$\sum_{i=1}^n\omega_i\log X^{-\frac12}A_iX^{-\frac12}=0 \tag{1.1}$$

for given n positive definite matrices $A_1,\cdots,A_n$, where $\omega=(\omega_1,\cdots,\omega_n)$ is a weight vector, i.e., $\omega_1,\cdots,\omega_n\ge0$ and $\sum_{i=1}^n\omega_i=1$. We say the solution X of (1.1) the *Karcher mean*, or the *Riemannian mean* for n positive definite matrices $A_1,\cdots,A_n$ and denote it by $G_k(\omega;A_1,\cdots,A_n)$, see also [9, 20]. In particular, in the case of $\omega=(\frac1n,\cdots,\frac1n)$ we denote it by $G_k(A_1,\cdots,A_n)$. In the case of $n=2$, the Karcher mean $G((1-\alpha,\alpha);A,B)$ coincides with the matrix geometric mean $A\ \sharp_\alpha\ B$. The matrix geometric mean $A\ \sharp_\alpha\ B$ satisfies the following Ando-Hiai inequality [1]: For $\alpha\in[0,1]$

$$A\ \sharp_\alpha\ B\le I\qquad\text{implies}\qquad A^p\ \sharp_\alpha\ B^p\le I\quad\text{for all } p\ge1.$$

Yamazaki [25] showed that the Karcher mean satisfies the n-variable Ando-Hiai inequality, though the ALM mean does not satisfy it. In [4], Bhagwat and Subramanjian showed that for positive definite $A_1,\cdots,A_n$ and a weight vector $\omega=(\omega_1,\cdots,\omega_n)$

$$\lim_{p\to0}\left(\sum_{i=1}^n\omega_iA_i^p\right)^{\frac1p}=\exp\left(\sum_{i=1}^n\omega_i\log A_i\right).$$

By taking the logarithm of the arithmetic-geometric-harmonic mean inequality, it follows that

$$\lim_{p\to0}G_k(\omega;A_1^p,\cdots,A_n^p)^{\frac1p}=\exp\left(\sum_{i=1}^n\omega_i\log A_i\right), \tag{1.2}$$

also see [12]. The right-hand side of (1.2) is called the *chaotic geometric mean* [14, 22, 23], or the Log-Euclidean mean [7, 3] and we denote it by

$$\diamondsuit(\omega;A_1,\cdots,A_n)\equiv\exp\left(\sum_{i=1}^n\omega_i\log A_i\right).$$

In particular, we denote $\diamondsuit(A_1,\cdots,A_n)=\exp\left(\frac1n\sum_{i=1}^n\log A_i\right)$ and $A\ \diamondsuit_\alpha\ B=\exp((1-\alpha)\log A+\alpha\log B)$ for $\alpha\in[0,1]$. The chaotic geometric mean does not have either of the properties (i) monotonicity and (ii) transformer equality. In fact, it is known that the exponential map is not order-preserving under the usual order. However, the chaotic geometric mean is monotone under the chaotic order and the arithmetic-geometric-harmonic mean inequality holds under the chaotic order, see [23]. Therefore, the chaotic geometric mean plays an important role in the field of means.

In this paper, from the viewpoint of the Ando-Hiai inequality, we make a comparison among three geometric means: The ALM mean, the Karcher mean and the chaotic geometric mean of n positive definite matrices. We show complements of the n-variable Ando-Hiai inequality for the ALM mean by means of the Kantorovich constant.

2. Preliminary

A norm $\|\cdot\|$ on $\mathbb{M}_d$ is said to be *unitarily invariant* if $\|UXV\| = \|X\|$ for all $X \in \mathbb{M}_d$ and all unitary U, V. We denote by $\| A \|_\infty$ the spectral (operator) norm of A: $\| A \|_\infty \equiv \max\{ \| Ax \| : \| x \| = 1, x \in \mathbb{C}^d\}$. For a Hermitian matrix $A \in \mathbb{M}_d$, we denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_d(A)$ the eigenvalues of A arranged in the decreasing order with their multiplicities counted. The notion $\lambda(A)$ stands for the row vector $(\lambda_1(A), \lambda_2(A), \cdots, \lambda_d(A))$. The eigenvalue inequality $\lambda(A) \leq \lambda(B)$ means $\lambda_j(A) \leq \lambda_j(B)$ for all $j = 1, \ldots, d$. For two Hermitian matrices A, B the inequality $\lambda(A) \leq \lambda(B)$ if and only if $A \leq UBU^*$ for some unitary matrix U. The weak majorization $\lambda(A) \prec_w \lambda(B)$ means $\sum_{i=1}^k \lambda_i(A) \leq \sum_{i=1}^k \lambda_i(B)$ for all $k = 1, \ldots, d$. It is known that $A \leq B \Longrightarrow \lambda(A) \leq \lambda(B) \Longrightarrow \lambda(A) \prec_w \lambda(B)$. The Ky Fan dominance theorem states that $\lambda(A) \prec_w \lambda(B)$ if and only if $\|A\| \leq \|B\|$ for positive semidefinite A and B. For more information on matrix analysis, see [5].

3. Specht type theorem

Specht [24] estimated the upper boundary of the arithmetic mean by the geometric one for positive numbers: For $a_1, \cdots, a_n \in [m, M]$ with $0 < m \leq M$

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n} \leq S(h) \sqrt[n]{a_1 a_2 \cdots a_n} \tag{3.1}$$

where $h = \frac{M}{m}$ and the *Specht ratio* $S(h)$ is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \ (h \neq 1) \quad \text{and} \quad S(1) = 1, \tag{3.2}$$

see also [15]. Therefore the Specht theorem (3.1) means a ratio type reverse inequality of the arithmetic-geometric mean inequality.

In [11], we showed the following Specht type theorem for the ALM mean: For positive definite $A_1, \cdots, A_n \in \mathbb{P}$ such that $mI \leq A_i \leq MI$ for $i = 1, \ldots, n$ and some scalars $0 < m \leq M$

$$G_{\mathrm{alm}}(A_1, \cdots, A_n) \leq \frac{1}{n} \sum_{i=1}^n A_i \leq \frac{(M+m)^2}{4Mm} G_{\mathrm{alm}}(A_1, \cdots, A_n), \tag{3.3}$$

where the constant $\frac{(M+m)^2}{4Mm}$ is called the *Kantorovich constant*. Though the weighted arithmetic-geometric mean inequality does not hold for the chaotic geometric mean, we showed the following inequality in [11, Lemma 12]: For a weight vector $\omega = (\omega_1, \cdots, \omega_n)$

$$S(h)^{-1} \diamondsuit(\omega; A_1, \cdots, A_n) \leq \sum_{i=1}^n \omega_i A_i \leq S(h) \diamondsuit(\omega; A_1, \cdots, A_n) \tag{3.4}$$

where $h = \frac{M}{m}$. Here, we state the relation between the Kantorovich constant and the Spehct ratio:

Lemma 3.1. *For $0 < m \le M$ and $h = \frac{M}{m}$*

$$S(h) \le \frac{(M+m)^2}{4Mm} \le S(h)^2. \tag{3.5}$$

Proof. The first inequality is due to [26]. For the second inequality, it follows from the definition of the Specht ratio that

$$\frac{m+M}{2} \le S(h)\sqrt{Mm}$$

and hence we have $\frac{(M+m)^2}{4Mm} \le S(h)^2$. □

We show the following Specht type theorem for the Karcher mean.

Theorem 3.2. *Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \dots, n$ and some scalars $0 < m < M$, and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. Then*

$$G_k(\omega; A_1, \cdots, A_n) \le \sum_{i=1}^n \omega_i A_i \le \frac{(M+m)^2}{4Mm} G_k(\omega; A_1, \cdots, A_n). \tag{3.6}$$

Proof. By the Kantorovich inequality [13], we have

$$\sum_{i=1}^n \omega_i A_i \le \frac{(M+m)^2}{4Mm} \left(\sum_{i=1}^n \omega_1 A_i^{-1} \right)^{-1}.$$

Since the Karcher mean satisfies the arithmetic-geometric-harmonic mean inequality, it follows that

$$\sum_{i=1}^n \omega_i A_i \le \frac{(M+m)^2}{4Mm} \left(\sum_{i=1}^n \omega_i A_i^{-1} \right)^{-1} \le \frac{(M+m)^2}{4Mm} G_k(\omega; A_1, \cdots, A_n).$$

□

Remark 3.3. Since the right hand side of (3.4) implies a commutative case (3.1), the inequality (3.4) is sharp. However, we don't know whether it is possible to replace the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ by the Specht ratio $S(h)$ in (3.3) and (3.6).

As a corollary, we have the following order relation between the Karcher mean and the chaotic geometric mean.

Corollary 3.4. *Let $A_1, \cdots, A_n \in \mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \dots, n$ and some scalars $0 < m < M$, and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. Put $h = \frac{M}{m}$. Then*

$$S(h)^{-1} \diamondsuit(\omega; A_1, \cdots, A_n) \le G_k(\omega; A_1, \cdots, A_n) \le S(h) \diamondsuit(\omega; A_1, \cdots, A_n).$$

Proof. By (3.4), we have

$$G_k(\omega; A_1, \cdots, A_n) \le \sum_{i=1}^n \omega_i A_i \le S(h) \diamondsuit(\omega; A_1, \cdots, A_n)$$

and it follows from the self duality of the Karcher mean and the chaotic geometric mean that

$$G_k(\omega; A_1, \cdots, A_n) \geq \left(\sum_{i=1}^{n} \omega_i A_i^{-1} \right)^{-1} \geq S(h)^{-1} \diamondsuit(\omega; A_1, \cdots, A_n).$$

□

Corollary 3.5. *Let $A_1, \cdots, A_n \in \mathbb{P}$ such that $mI \leq A_i \leq MI$ for $i = 1, \ldots, n$ and some scalars $0 < m < M$, and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. Put $h = \frac{M}{m}$. Then*

$$|||G_k(\omega; A_1, \cdots, A_n)||| \leq |||\diamondsuit(\omega; A_1, \cdots, A_n)||| \leq S(h)\, |||G_k(\omega; A_1, \cdots, A_n)|||$$

for every unitarily invariant norm $|||\cdot|||$, where the Specht ratio $S(h)$ is defined by (3.2).

Proof. The first inequality is due to Theorem C in §4 and (1.2). The second inequality is due to Corollary 3.4. □

Similarly we have the following order relation between the ALM mean and the chaotic geometric mean.

Corollary 3.6. *Let $A_1, \cdots, A_n \in \mathbb{P}$ such that $mI \leq A_i \leq MI$ for $i = 1, \ldots, n$ and some scalars $0 < m < M$, and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. Put $h = \frac{M}{m}$. Then*

$$S(h)^{-1} \diamondsuit(\omega; A_1, \cdots, A_n) \leq G_{\mathrm{alm}}(\omega; A_1, \cdots, A_n) \leq S(h) \diamondsuit(\omega; A_1, \cdots, A_n)$$

and

$$S(h)^{-1}\, |||G_{\mathrm{alm}}(\omega; A_1, \cdots, A_n)||| \leq |||\diamondsuit(\omega; A_1, \cdots, A_n)||| \leq S(h)\, |||G_{\mathrm{alm}}(\omega; A_1, \cdots, A_n)|||$$

for every unitarily invariant norm $|||\cdot|||$, where the Specht ratio $S(h)$ is defined by (3.2).

4. Ando-Hiai inequality for the ALM geometric mean

In this section, from the viewpoint of the Ando-Hiai inequality, we make a comparison among three geometric means: The ALM mean, the Karcher mean and the chaotic geometric mean.

Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. By definition, the chaotic geometric mean satisfies the n-variable Ando-Hiai inequality: $\diamondsuit(\omega; A_1, \cdots, A_n) \leq I$ implies $\diamondsuit(\omega; A_1^p, \cdots, A_n^p) \leq I$ for all $p > 0$. On the other hand, Yamazaki [25] showed that

$$\sum_{i=1}^{n} \omega_i \log A_i \leq 0 \qquad \text{implies} \qquad G_k(\omega; A_1, \cdots, A_n) \leq I. \tag{4.1}$$

By (4.1), Yamazaki showed the following n-variable Ando-Hiai inequality for the Karcher mean:

Theorem B. *Let $A_1, \cdots, A_n \in \mathbb{P}$ and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. Then*

$$G_k(\omega; A_1, \cdots, A_n) \leq I \quad \textit{implies} \quad G_k(\omega; A_1^p, \cdots, A_n^p) \leq I \quad \textit{for all } p \geq 1 \tag{4.2}$$

or equivalently,

$$\| \, G_k(\omega; A_1^p, \cdots, A_n^p) \, \|_\infty \leq \| \, G_k(\omega; A_1, \cdots, A_n)^p \, \|_\infty \quad \textit{for all } p \geq 1.$$

By Theorem B, Hiai [16] showed the following log-majorization for the Karcher mean:

Theorem C. *Let $A_1, \cdots, A_n \in \mathbb{P}$ and a weight vector $\omega = (\omega_1, \cdots, \omega_n)$. Then for $p \geq q > 0$*

$$|||G_k(\omega; A_1^p, \cdots, A_n^p)^{1/p}||| \leq |||G_k(\omega; A_1^q, \cdots, A_n^q)^{1/q}|||$$

for every unitarily invariant norm $|||\cdot|||$.

If the ALM mean satisfies the n-variable Ando-Hiai inequality, then it follows from [25, Corollary 6] that the ALM mean coincides with the Karcher mean. It is known that the ALM mean is different from the Karcher mean. Hence the ALM mean doe not satisfy the n-variable Ando-Hiai inequality. Then we have the following evaluation among three geometric means by means of the Kantorovich constant:

Theorem 4.1. *Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ such that $mI \leq A_i \leq MI$ for $i = 1, \ldots, n$ and some scalars $0 < m < M$. Then the following assertions are mutually equivalent:*

(i) $\diamondsuit(A_1, \cdots, A_n) \leq I$.

(ii) $G_k(A_1^p, \cdots, A_n^p) \leq I$ *for all* $p > 0$.

(iii) $G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p) \leq \frac{(M^p+m^p)^2}{4M^p m^p} I$ *for all* $p > 0$.

Proof. The equivalence of (i) and (ii) is shown in [25, Theorem 4].

Proof of (ii)$\Longrightarrow$(iii). By the Kantorovich inequality [13], we have

$$\begin{aligned} G_{\mathrm{alm}}(A_1, \cdots, A_n) &\leq \frac{1}{n}\sum_{i=1}^n A_i \leq \frac{(M+m)^2}{4Mm}\left(\frac{1}{n}\sum_{i=1}^n A_i^{-1}\right)^{-1} \\ &\leq \frac{(M+m)^2}{4Mm} G_k(A_1, \cdots, A_n) \end{aligned}$$

and hence it follows from $m^p I \leq A_i^p \leq M^p I$ for $i = 1, \ldots, n$ that

$$G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p) \leq \frac{(M^p+m^p)^2}{4M^p m^p} G_k(A_1^p, \cdots, A_n^p) \leq \frac{(M^p+m^p)^2}{4M^p m^p} I$$

for all $p > 0$.

Proof of (iii)$\Longrightarrow$(i). By assumption

$$G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \leq \left(\frac{(M^p+m^p)^2}{4M^p m^p}\right)^{\frac{1}{p}} I$$

for all $p > 0$. Since $\lim_{p\to 0}\left(\frac{(M^p+m^p)^2}{4M^p m^p}\right)^{\frac{1}{p}} = 1$ and the ALM mean $G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}}$ converges to the chaotic geometric mean $\diamondsuit(A_1, \cdots, A_n)$ as $p \to 0$ in [12], we have (i). □

We recall the definition of the *generalized Kantorovich constant*, also see [15, Definition 2.2]. For positive definite $A_1, \cdots, A_n$ in $\mathbb{P}$, Jensen operator inequality for an operator convex function says that

$$\left(\frac{1}{n}\sum_{i=1}^n A_i\right)^p \leq \frac{1}{n}\sum_{i=1}^n A_i^p \qquad \text{for all } 1 \leq p \leq 2.$$

Then we have the following reverse of Jensen operator inequality:

$$\frac{1}{n}\sum_{i=1}^{n} A_i^p \le K(m,M,p)\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right)^p \qquad \text{for all } p \ge 1, \tag{4.3}$$

where the generalized Kantorovich constant $K(m,M,p)$ is defined by

$$K(m,M,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p \tag{4.4}$$

for any real numbers $p \in \mathbb{R}$. In particular, $K(m,M,2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$.

We need the following lemma to show our results.

Lemma 4.2. *Let A and B be positive definite matrices. If $A \le B$, then there exists a unitary matrix U such that $A^p \le UB^pU^*$ for all $p > 0$.*

Proof. If $A \le B$, then there exists a unitary matrix U such that $A \le UBU^*$ and A commutes with UBU^*. Hence we have $A^p \le (UBU^*)^p = UB^pU^*$ for all $p > 0$. □

By Theorem 3.2, we have the following complements of the n-variable Ando-Hiai inequality for the ALM mean.

Theorem 4.3. *Let $A_1, \cdots, A_n$ be positive definite marices in $\mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \ldots, n$ and some scalars $0 < m < M$. Then*

$$\| G_{\text{alm}}(A_1^p, \cdots, A_n^p) \| \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p \| G_{\text{alm}}(A_1, \cdots, A_n)^p \| \tag{4.5}$$

for all $p \ge 1$ and every unitarily invariant norm $\|\cdot\|$. In particular,

$$G_{\text{alm}}(A_1, \cdots, A_n) \le I \text{ implies } G_{\text{alm}}(A_1^p, \cdots, A_n^p) \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p I \tag{4.6}$$

for all $p \ge 1$.

Proof. By the arithmetic-geometric mean inequality and (4.3), it follows that

$$G_{\text{alm}}(A_1^p, \cdots, A_n^p) \le \frac{1}{n}\sum_{i=1}^{n} A_i^p \le K(m,M,p)\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right)^p \quad \text{for all } p \ge 1.$$

On the other hand, by the Specht type theorem for the ALM mean, it follows that

$$\frac{1}{n}\sum_{i=1}^{n} A_i \le \frac{(M+m)^2}{4Mm} G_{\text{alm}}(A_1, \cdots, A_n)$$

and hence by Lemma 4.2 there exists a unitary matrix U such that

$$\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right)^p \le \left(\frac{(M+m)^2}{4Mm}\right)^p UG_{\text{alm}}(A_1, \cdots, A_n)^p U^*$$

for all $p \ge 1$. Combining two inequalities above, we have

$$G_{\text{alm}}(A_1^p, \cdots, A_n^p) \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p UG_{\text{alm}}(A_1, \cdots, A_n)^p U^*$$

and this implies

$$|||G_{\rm alm}(A_1^p,\cdots,A_n^p)||| \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p |||G_{\rm alm}(A_1,\cdots,A_n)^p||| \quad \text{for all } p\ge 1.$$

In particular, it follows that

$$\| G_{\rm alm}(A_1^p,\cdots,A_n^p) \|_\infty \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p \| G_{\rm alm}(A_1,\cdots,A_n)^p \|_\infty$$

for all $p \ge 1$. If $G_{\rm alm}(A_1,\cdots,A_n)\le I$, then

$$G_{\rm alm}(A_1^p,\cdots,A_n^p) \le \| G_{\rm alm}(A_1^p,\cdots,A_n^p) \|_\infty \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p$$

for all $p \ge 1$. □

Remark 4.4. We can improve the result of (4.6) a little more. In fact, it follows that

$$G_{\rm alm}(A_1^p,\cdots,A_n^p) \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^{p-1} UG_{\rm alm}(A_1,\cdots,A_n)^{p-1}U^*\left(\frac{1}{n}\sum_{i=1}^n A_i\right)$$

and hence if $G_{\rm alm}(A_1,\cdots,A_n)\le I$, then we have

$$G_{\rm alm}(A_1^p,\cdots,A_n^p) \le K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^{p-1}\left(\frac{1}{n}\sum_{i=1}^n A_i\right)$$

for all $p\ge 1$. Put

$$\Delta(p)=\min\{K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^p, K(m,M,p)\left(\frac{(M+m)^2}{4Mm}\right)^{p-1} M\}$$

for $p\ge 1$. Then we have the following Ando-Hiai inequality for the ALM mean.

$$G_{\rm alm}(A_1,\cdots,A_n)\le I \quad \text{implies} \quad G_{\rm alm}(A_1^p,\cdots,A_n^p)\le \Delta(p)I$$

for all $p\ge 1$. However, we have $\Delta(1)\ne 1$.

Theorem 4.5. *Let $A_1,\cdots,A_n$ be positive definite marices in $\mathbb{P}$ such that $mI\le A_i\le MI$ for $i=1,\dots,n$ and some scalars $0<m<M$. Then*

$$G_{\rm alm}(A_1^p,\cdots,A_n^p) \le \left(\frac{(M+m)^2}{4Mm}\right)^p G_{\rm alm}(A_1,\cdots,A_n)^p \quad \text{for all } 0<p<1. \tag{4.7}$$

In particular,

$$G_{\rm alm}(A_1,\cdots,A_n)\le I \quad \text{implies} \quad G_{\rm alm}(A_1^p,\cdots,A_n^p)\le \left(\frac{(M+m)^2}{4Mm}\right)^p I \tag{4.8}$$

for all $0<p<1$.

Proof. By the arithmetic-geometric mean inequality and $0<p<1$, it follows from the Löwner-Heinz theorem and (3.3) that

$$G_{\rm alm}(A_1^p,\cdots,A_n^p) \le \frac{1}{n}\sum_{i=1}^n A_i^p \le \left(\frac{1}{n}\sum_{i=1}^n A_i\right)^p \le \left(\frac{(M+m)^2}{4Mm}\right)^p G_{\rm alm}(A_1,\cdots,A_n)^p$$

for all $0<p<1$. □

By using the Specht type theorem for the ALM mean, we have the following complements of the log-majorization for the ALM mean.

Theorem 4.6. *Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \ldots, n$ and some scalars $0 < m \le M$. Then for each $0 < q \le p$*

$$K(m^q, M^q, \frac{p}{q})^{-\frac{1}{p}} \left(\frac{4M^q m^q}{(M^q+m^q)^2}\right)^{\frac{1}{q}} \left\| G_{\text{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right\| \le \left\| G_{\text{alm}}(A_1^q, \cdots, A_n^q)^{\frac{1}{q}} \right\|$$

$$\le K(m^q, M^q, \frac{p}{q})^{\frac{1}{p}} \left(\frac{(M^q+m^q)^2}{4M^q m^q}\right)^{\frac{1}{q}} \left\| G_{\text{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right\|$$

for every unitarily invariant norm $\|\cdot\|$, where the generalized Kantorovich constant $K(m, M, p)$ is defined by (4.4)*.*

Proof. For each $0 < q \le p$, it follows from the arithmetic-geometric mean inequality and (4.3) that

$$G_{\text{alm}}(A_1^{\frac{p}{q}}, \cdots, A_n^{\frac{p}{q}}) \le \frac{1}{n}\sum_{i=1}^{n} A_i^{\frac{p}{q}} \le K(m, M, \frac{p}{q}) \left(\frac{1}{n}\sum_{i=1}^{n} A_i\right)^{\frac{p}{q}} \quad \text{by } \tfrac{p}{q} \ge 1.$$

Replacing A_i by A_i^q, we have

$$G_{\text{alm}}(A_1^p, \cdots, A_n^p) \le K(m^q, M^q, \frac{p}{q}) \left(\frac{1}{n}\sum_{i=1}^{n} A_i^q\right)^{\frac{p}{q}}.$$

On the other hand, by (3.3) and $m^q I \le A_i^q \le M^q I$ for $i = 1, \ldots, n$ we have

$$\frac{1}{n}\sum_{i=1}^{n} A_i^q \le \frac{(M^q+m^q)^2}{4M^q m^q} G_{\text{alm}}(A_1^q, \cdots, A_n^q)$$

and by Lemma 4.2 that there exists a unitary matrix U such that

$$\left(\frac{1}{n}\sum_{i=1}^{n} A_i^q\right)^{\frac{p}{q}} \le \left(\frac{(M^q+m^q)^2}{4M^q m^q}\right)^{\frac{p}{q}} U G_{\text{alm}}(A_1^q, \cdots, A_n^q)^{\frac{p}{q}} U^*.$$

Therefore it follows that

$$G_{\text{alm}}(A_1^p, \cdots, A_n^p) \le K(m^q, M^q, \frac{p}{q}) \left(\frac{(M^q+m^q)^2}{4M^q m^q}\right)^{\frac{p}{q}} U G_{\text{alm}}(A_1^q, \cdots, A_n^q)^{\frac{p}{q}} U^*.$$

By Lemma 4.2 again, there exists a unitary matrix V such that

$$G_{\text{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \le K(m^q, M^q, \frac{p}{q})^{\frac{1}{p}} \left(\frac{(M^q+m^q)^2}{4M^q m^q}\right)^{\frac{1}{q}} V G_{\text{alm}}(A_1^q, \cdots, A_n^q)^{\frac{1}{q}} V^*$$

and this implies the first inequality of the desired one. Replacing A_i by A_i^{-1} in the inequality above and moreover taking the inverse, it follows from the self-duality of the ALM mean that

$$G_{\text{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \ge \left(K(m^q, M^q, \frac{p}{q})^{\frac{1}{p}} \left(\frac{(M^q+m^q)^2}{4M^q m^q}\right)^{\frac{1}{q}}\right)^{-1} V G_{\text{alm}}(A_1^q, \cdots, A_n^q)^{\frac{1}{q}} V^*$$

and this implies the second inequality of the desired one. ◻

If we put $q \to 0$ in Theorem 4.6, then we have the following theorem.

Theorem 4.7. *Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \ldots, n$ and some scalars $0 < m \le M$. Put $h = \frac{M}{m}$. Then*

$$S(h^p)^{-\frac{1}{p}} \left|\!\left|\!\left| G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right|\!\right|\!\right| \le |||\Diamond(A_1, \cdots, A_n)||| \le S(h^p)^{\frac{1}{p}} \left|\!\left|\!\left| G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right|\!\right|\!\right|$$

for all $p > 0$ and every unitarily invariant norm $|||\cdot|||$. In particular,

$$S(h^p)^{\frac{1}{p}} \left|\!\left|\!\left| G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right|\!\right|\!\right| \to |||\Diamond(A_1, \cdots, A_n)||| \quad \text{as } p \to 0,$$

where the Specht ratio $S(h)$ is defined by (3.2).

Proof. It follows from $K(m^q, M^q, \frac{p}{q})^{\frac{1}{p}} \to S(h^p)^{\frac{1}{p}}$ as $q \to 0$ in [15, Theorem 2.56]. □

Remark 4.8. If we put $p = 1$ in Theorem 4.7, then we have Corollary 3.6.

5. Comparisons

In this final section, we make a comparison between the ALM mean and the Karcher mean. Though the ALM mean does not coincide with the Karcher one in general, we have the following estimate.

Theorem 5.1. *Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \ldots, n$ and some scalars $0 < m \le M$. Then*

$$\frac{4Mm}{(M+m)^2} G_k(A_1, \cdots, A_n) \le G_{\mathrm{alm}}(A_1, \cdots, A_n) \le \frac{(M+m)^2}{4Mm} G_k(A_1, \cdots, A_n).$$

Proof. The second inequality follows from the proof of (ii)$\Longrightarrow$(iii) of Theorem 4.1. For the first inequality, we similarly have

$$G_k(A_1, \cdots, A_n) \le \frac{1}{n}\sum_{i=1}^n A_i \le \frac{(M+m)^2}{4Mm}\left(\frac{1}{n}\sum_{i=1}^n A_i^{-1}\right)^{-1} \le \frac{(M+m)^2}{4Mm} G_{\mathrm{alm}}(A_1, \cdots, A_n).$$

□

By Theorem 5.1, we have the following norm inequality.

Theorem 5.2. *Let $A_1, \cdots, A_n$ be positive definite matrices in $\mathbb{P}$ such that $mI \le A_i \le MI$ for $i = 1, \ldots, n$ and some scalars $0 < m \le M$. Then*

$$\left(\frac{4M^pm^p}{(M^p+m^p)^2}\right)^{\frac{1}{p}} \left|\!\left|\!\left| G_k(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right|\!\right|\!\right| \le \left|\!\left|\!\left| G_{\mathrm{alm}}(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right|\!\right|\!\right| \le \left(\frac{(M^p+m^p)^2}{4M^pm^p}\right)^{\frac{1}{p}} \left|\!\left|\!\left| G_k(A_1^p, \cdots, A_n^p)^{\frac{1}{p}} \right|\!\right|\!\right|$$

for all $p > 0$ and every unitarily invariant norm $|||\cdot|||$.

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