OPERATOR POWER MEANS DUE TO LAWSON-LIM-PÁLFIA FOR $1 < t < 2$

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ABSTRACT. For $-1 \le t \le 1$, Lim-Pálfia defined a new family of operator power means of positive definite matrices and subsequently by Lawson-Lim their notion and most of their results extend to the setting of positive invertible operators on a Hilbert space. Each of these means except $t \neq 0$ arises as a unique positive invertible solution of a non-linear operator equation and satisfies all desirable properties of power arithmetic means of positive real numbers. The purpose of this paper is to extend the range in which operator power means due to Lawson-Lim-Pálfia are defined. We investigate some properties of operator power means for $t \in (-2,2)\backslash[-1,1]$.

1. INTRODUCTION

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm, $S(H)$ the set of all bounded self-adjoint operators, and $\mathbb{P} = \mathbb{P}(H)$ the open convex cone of all positive invertible operators. For $X, Y \in S(H)$, we write $X \le Y$ if $Y - X$ is positive, and $X < Y$ if $Y - X$ is positive invertible.

For positive real numbers $x_1, \dots, x_n \in \mathbb{R}$ and a weight vector $\omega = (\omega_1, \dots, \omega_n)$ such as $\omega_i \ge 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^n \omega_i = 1$, the power arithmetic means

$$M_t(\omega; x_1, \dots, x_n) = \left(\omega_1 x_1^t + \cdots + \omega_n x_n^t\right)^{1/t} \quad \text{for } t \in \mathbb{R}$$

make a path of means from the harmonic one at $t = -1$ to the arithmetic one at $t = 1$ via the geometric one at $t \to 0$. The following is a noncommutative version of the power arithmetic mean: For positive invertible operators $A_1, \dots, A_n \in \mathbb{P}$ and a weight vector ω

$$M_t(\omega; A_1, \dots, A_n) = \left(\sum_{i=1}^n \omega_i A_i^t\right)^{1/t} \qquad \text{for } t \in \mathbb{R}.$$

Bhagwat and Subramanian [2] showed that the power arithmetic mean has the following monotonicity:

$$1 \le t \le s \quad \Longrightarrow \quad M_t(\omega; A_1, \dots, A_n) \le M_s(\omega; A_1, \dots, A_n)$$

and the limit $M_0(\omega; A_1, \dots, A_n) = u\text{-}\lim_{t \to 0+} M_t(\omega; A_1, \dots, A_n)$ exists and is equal to the chaotic geometric mean $\exp(\sum_{i=1}^n \omega_i \log A_i)$, also see [7, 15], which reduced to the usual geometric mean in the case of commuting operators. However, $M_t(\omega; A_1, \dots, A_n)$ does not have the monotonicity for $-1 < t < 1$ in general and they are not operator means except for the case of $t = \pm 1$.

Recently, for $-1 \le t \le 1$, Lim-Pálfia [13] defined a new family of operator power means of positive definite matrices and subsequently by Lawson-Lim [12] their notion and most of their results extend to the setting of positive invertible operators on a Hilbert space.

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We denote by $\{P_t(\omega;\mathbb{A})\}$, where ω is a weight vector and $\mathbb{A}$ is an n-tuple of positive invertible operators on a Hilbert space. Each of these means except $t \neq 0$ arises as a unique positive invertible solution $P_t(\omega;\mathbb{A})$ of a non-linear operator equation

$$X = \sum_{i=1}^{n} \omega_i (X \sharp_t A_i) \qquad (t \in [-1,1]\backslash\{0\})$$

and satisfies desirable properties of power arithmetic means of positive real numbers and interpolates between the weighted harmonic and arithmetic means. Moreover, Lawson-Lim showed that the Karcher mean of positive invertible operators coincides with the limit of operator power means as $t \to 0$. For more details on the Karcher mean; see [4, 5, 17]. In fact, if A_i mutually commute for $i = 1,\ldots,n$, then it follows that $P_t(\omega;\mathbb{A}) = (\sum_{i=1}^{n} \omega_i A_i^t)^{1/t}$. Moreover, they showed that the power means $P_t(\omega;\mathbb{A})$ have a monotone increasing property for $-1 < t < 1$:

$$-1 < t \le s < 1 \quad \Longrightarrow \quad P_t(\omega;\mathbb{A}) \le P_s(\omega;\mathbb{A})$$

and an information monotonicity:

$$\Phi(P_t(\omega;\mathbb{A})) \le P_t(\omega;\Phi(\mathbb{A})) \qquad (t \in (0,1])$$

for any unital positive limear map Φ.

However, the range in which the operator power means are defined, is limited to $[-1,1]$. The purpose of this paper is to extend the range of the definition of power means $P_t(\omega;\mathbb{A})$. Moreover, we investigate some properties of $P_t(\omega;\mathbb{A})$ for $t \in (-2,2)\backslash[-1,1]$.

2. Preliminaries

For $A, B \in \mathbb{P}$ and $t \in [0,1]$, the t-geometric operator mean is defined as

$$A \sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}.$$

For convenience, we use the notation $\natural_t$ for the binary operation

$$A \natural_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2} \qquad \text{for } t \notin [0,1],$$

whose formula is the same as $\sharp_t$. Though $A \sharp_t B$ for $t \in [0,1]$ is monotonic, $A \natural_s B$ for $s \notin [0,1]$ is not.

Lemma 2.1. *Let $A, B, X, Y \in \mathbb{P}$ and $1 < t \le 2$. Then*
(i) *If $X \le Y$, then $Y \natural_t A \le X \natural_t A$.*
(ii) *If $A \le B$ with $m_1 \le A \le M_1$, $m_2 \le B \le M_2$ and $m \le X \le M$ for some scalars $0 < m_i \le M_i$ $(i = 1,2)$ and $0 < m \le M$, then*

$$X \natural_t A \le K(m_i/M, M_i/m, t) X \natural_t B \quad \textit{for } i = 1,2,$$

where the generalized Kantorovich constant $K(m, M, t)$ is defined by

$$K(m,M,t) = \frac{mM^t - Mm^t}{(t-1)(M-m)}\left(\frac{t-1}{t}\frac{M^t - m^t}{mM^t - Mm^t}\right)^t \tag{1}$$

for any real number $t \in \mathbb{R}$, see [11, Theorem 2.53].
(iii) *If $m \le A \le M$ for some scalars $0 < m \le M$, then*

$$\| X \|^{1-t} m^t \le X \natural_t A \le \| X^{-1} \|^{-(1-t)} M^t.$$

Proof. (i): For $1 < t \le 2$

$$\begin{aligned} Y \natural_t A = A \natural_{1-t} Y &= A^{1/2}\left(A^{-1/2}YA^{-1/2}\right)^{1-t}A^{1/2} \\ &= A^{1/2}\left(A^{1/2}Y^{-1}A^{1/2}\right)^{t-1}A^{1/2} \\ &\le A^{1/2}\left(A^{1/2}X^{-1}A^{1/2}\right)^{t-1}A^{1/2} \quad \text{by } 0<t-1<1 \text{ and } Y^{-1} \le X^{-1} \\ &= X \natural_t A. \end{aligned}$$

(ii): Since $A \le B$, we have $X^{-1/2}AX^{-1/2} \le X^{-1/2}BX^{-1/2}$ and $m_1/M \le X^{-1/2}AX^{-1/2} \le M_1/m$ and $m_2/M \le X^{-1/2}BX^{-1/2} \le M_2/m$. By the generalized Kantorovich inequality [11, Theorem 8.3], it follows from $1 < t \le 2$ that

$$(X^{-1/2}AX^{-1/2})^t \le K(\frac{m_i}{M}, \frac{M_i}{m}, t)(X^{-1/2}BX^{-1/2})^t \quad \text{for } i = 1, 2,$$

and we have the desired inequality.
(iii): It follows from $\| X^{-1} \|^{-1} \le X \le \| X \|$ and (i). □

Remark 2.2. Let $X \ge 0$ and $0 < A \le B$. Then the inequality $AXA \le BXB$ doe not hold in general. If we put $t = 2$ in (2) of Lemma 2.1, then we have

$$AXA \le \min\left\{\frac{(nm_1 + NM_1)^2}{4nNm_1M_1}, \frac{(nm_2 + NM_2)^2}{4nNm_2M_2}\right\} BXB$$

where $n \le X \le N$ and $m_1 \le A \le M_1, m_2 \le B \le M_2$ for some scalars $0 < n \le N$ and $0 < m_i \le M_i$ $(i = 1, 2)$. If $B = I$, then we have $AXA \le \frac{(n+N)^2}{4nN}X$ in [10, Lemma 4].

3. Thompson metric

The Thompson metric on $\mathbb{P}$ is defined by

$$d(A, B) = \log \max\{M(A/B), M(B/A)\}$$

where $M(A/B) = \inf\{\lambda > 0 : A \le \lambda B\} = \| B^{-1/2}AB^{-1/2} \| = r(B^{-1}A)$. It is known that d is a complete metric on $\mathbb{P}$ and

$$d(A, B) = \| \log B^{-1/2}AB^{-1/2} \| = \| \log A^{-1/2}BA^{-1/2} \|,$$

see [16]. We list some basic properties of the Thompson metric:

Lemma 3.1 (see [3, 6])**.** *For $A, B, C, D \in \mathbb{P}$*

(i) $d(A, B) = d(A^{-1}, B^{-1}) = d(T^*AT, T^*BT)$ *for invertible* $T \in B(H)$*;*
(ii) $d(A + B, C + D) \le \max\{d(A, C), d(B, D)\}$*;*
(iii) $d(A^t, B^t) \le td(A, B)$ *for* $t \in [0, 1]$*;*
(iv) $d(\alpha A, \alpha B) = d(A, B)$ *for positive real number* $\alpha > 0$*;*
(v) $d(A\sharp_t B, C\sharp_t D) \le (1 - t)d(A, C) + td(B, D)$ *for* $t \in [0, 1]$*.*

For $A, B \in \mathbb{P}$, a map $\gamma_{A,B} : \mathbb{R} \mapsto \mathbb{P}$ defined by $\gamma_{A,B}(t) = A\natural_t B$ for $t \in \mathbb{R}$ is a path joining A and B. Then we have the following:

Theorem 3.2. *Let $A, B \in \mathbb{P}$. Then*

$$d(A\natural_s B, A\natural_t B) = |s - t|d(A, B) \qquad \textit{for all } s, t \in \mathbb{R}.$$

Proof. By definition of the Thompson metric and Lemma 3.1

$$\begin{aligned} d(A\natural_s B, A\natural_t B) &= d((A^{-1/2}BA^{-1/2})^s, (A^{-1/2}BA^{-1/2})^t) \\ &= d((A^{-1/2}BA^{-1/2})^{s-t}, I) = \| \log(A^{-1/2}BA^{-1/2})^{s-t} \| \\ &= |s-t| \, \| \log A^{-1/2}BA^{-1/2} \| = |s-t| d(A,B). \end{aligned}$$

□

We have the following estimate in the case of $1 < t < 2$, which corresponds to (v) of Lemma 3.1:

Theorem 3.3. *Let $A, B, C, D \in \mathbb{P}$ such that $m_1 A \le C \le M_1 A$ and $m_2 B \le D \le M_2 B$ for some scalars $0 < m_1 \le M_1$ and $0 < m_2 \le M_2$. For each $1 < t < 2$*

$$d(A\natural_t B, C\natural_t D) \le (t-1)d(A,C) + td(B,D) + \log K(t)$$

where $K(t) = \max\{K(m_1, M_1, t), K(m_2, M_2, t)\}$ and the generalized Kantorovich constant $K(m, M, t)$ is defined by (1)*.*

Proof. Since $\| A^{1/2}C^{-1}A^{1/2} \|^{-1} \le A^{-1/2}CA^{-1/2}$, it follows from Lemma 2.1 that

$$\begin{aligned} C \natural_t D &= A^{1/2} \left[(A^{-1/2}CA^{-1/2}) \natural_t (A^{-1/2}DA^{-1/2}) \right] A^{1/2} \\ &\le A^{1/2} \left[\| A^{1/2}C^{-1}A^{1/2} \|^{-1} \natural_t (A^{-1/2}DA^{-1/2}) \right] A^{1/2} \quad \text{by (i) of Lemma 2.1} \\ &= \| A^{1/2}C^{-1}A^{1/2} \|^{-(1-t)} A \natural_t D \\ &= \| A^{1/2}C^{-1}A^{1/2} \|^{-(1-t)} B^{1/2} \left[(B^{-1/2}AB^{-1/2}) \natural_t (B^{-1/2}DB^{-1/2}) \right] B^{1/2} \\ &\le \| A^{1/2}C^{-1}A^{1/2} \|^{-(1-t)} \| B^{-1/2}DB^{-1/2} \|^t K(m_2, M_2, t) B\natural_{1-t} A \quad \text{by (ii) of Lemma 2.1} \\ &= \| A^{1/2}C^{-1}A^{1/2} \|^{-(1-t)} \| B^{-1/2}DB^{-1/2} \|^t K(m_2, M_2, t) A\natural_t B. \end{aligned}$$

Similarly, it follows that

$$A\natural_t B \le \| C^{1/2}A^{-1}C^{1/2} \|^{t-1} \| D^{-1/2}BD^{1/2} \|^t K(m_1, M_1, t) \; C\natural_t D.$$

Therefore, we have

$$\| (A\natural_t B)^{-1/2}(C\natural_t D)(A\natural_t B)^{-1/2} \| \le \| A^{1/2}C^{-1}A^{1/2} \|^{-(1-t)} \| B^{-1/2}DB^{-1/2} \|^t K(m_2, M_2, t)$$

and

$$\| (C\natural_t D)^{-1/2}(A\natural_t B)(C\natural_t D)^{-1/2} \| \le \| C^{1/2}A^{-1}C^{1/2} \|^{t-1} \| D^{-1/2}BD^{1/2} \|^t K(m_1, M_1, t)$$

and this implies the desired inequality. □

4. Operator power means

In this section, we extend the range in which the power means due to Lawson-Lim-Pálfia are defined. For this, we need the following Lemma:

Lemma 4.1. *Let $X, Y, A \in \mathbb{P}$ and $1 < t \le 2$. Then*

$$d(X\natural_t A, Y\natural_t A) \le (t-1)d(X,Y).$$

Proof. For $1 < t \le 2$,

$$\begin{aligned} d(X\natural_t A, Y\natural_t A) &= d(A\natural_{1-t}X, A\natural_{1-t} Y) \\ &= d((A^{1/2}X^{-1}A^{1/2})^{t-1}, (A^{1/2}Y^{-1}A^{1/2})^{t-1}) \quad \text{by (i) of Lemma 3.1} \\ &\le (t-1)d(A^{1/2}XA^{1/2}, A^{1/2}YA^{1/2}) \quad \text{by (iii) of Lemma 3.1} \\ &= (t-1)d(X,Y) \quad \text{by (i) of Lemma 3.1.} \end{aligned}$$

□

Theorem 4.2. *Let $A_1, A_2, \ldots, A_n \in \mathbb{P}$ and a weight vector $\omega = (\omega_1, \ldots, \omega_n)$. Then for each $1 < t < 2$, the following equation has a unique positive invertible solution:*

$$X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i).$$

Proof. We will show that the map $f : \mathbb{P} \mapsto \mathbb{P}$ defined by $f(X) = \sum_{i=1}^{n} \omega_i(X\natural_t A_i)$ is a strict contraction with respect to the Thompson metric. Let $X, Y > 0$.

$$\begin{aligned} d(f(X), f(Y)) &\le \max_{1\le i\le n}\{d(\omega_i(X\natural_t A_i), \omega_i(Y\natural_t A_i))\} \quad \text{by (ii) of Lemma 3.1} \\ &= \max_{1\le i\le n}\{d(X\natural_t A_i, Y\natural_t A_i)\} \quad \text{by (iv) of Lemma 3.1} \\ &\le (t-1)d(X,Y) \quad \text{by Lemma 4.1.} \end{aligned}$$

Since $1 < t < 2$, it follows that f is a strict contraction and hence f has a unique fixed point. □

Definition 4.3. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ and a weight vector $\omega = (\omega_1, \ldots, \omega_n)$. For $t \in (1, 2)$, we denote by $P_t(\omega; \mathbb{A})$ the unique positive invertible solution of*

$$X = \sum_{i=1}^{n} \omega_i (X \natural_t A_i).$$

For $t \in (-2, -1)$, we define $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$. In fact, $X = P_t(\omega; \mathbb{A})$ is the unique positive invertible solution of $X = (\sum_{i=1}^{n} \omega_i(X\natural_{-t}A_i)^{-1})^{-1}$ and $X^{-1} = \sum_{i=1}^{n} \omega_i(X^{-1}\natural_{-t}A_i^{-1})$ if and only if $X^{-1} = P_{-t}(\omega; \mathbb{A}^{-1})$.

Remark 4.4. Let $t \in (1, 2)$. Put $f : \mathbb{P} \mapsto \mathbb{P}$ defined by $f(X) = \sum_{i=1}^{n} \omega_i(X\natural_t A_i)$. By Theorem 4.2, f is a strict contraction for the Thompson metric and by the Banach fixed point theorem

$$\lim_{k\to\infty} f^k(X) = P_t(\omega; \mathbb{A}) \quad \text{for any } X \in \mathbb{P}.$$

Similarly, the map $g(X) = (\sum_{i=1}^{n} \omega_i(X\natural_{-t}A_i)^{-1})^{-1}$ is a strict contraction for the Thompson metric and $\lim_{k\to\infty} g^k(X) = P_{-t}(\omega; \mathbb{A})$ for any $X \in \mathbb{P}$.

For $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n, M \in B(H), \omega = (\omega_1, \ldots, \omega_n)$ and for a permutation σ on n-letters, we set

$$M\mathbb{A}M^* = (MA_1M^*, \ldots, MA_nM^*), \qquad A_\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$$
$$\hat{\omega} = \frac{1}{1-\omega_n}(\omega_1, \ldots, \omega_{n-1}).$$

We list some basic properties of $P_t(\omega; \mathbb{A})$ for $t \in (-2, 2)\backslash[-1, 1]$.

Proposition 4.5. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$, a weight vector $\omega = (\omega_1, \ldots, \omega_n)$ and let $t \in (-2, 2)\backslash[-1, 1]$.*

(i) $P_t(\omega; \mathbb{A}) = (\sum_{i=1}^n \omega_i A_i^t)^{1/t}$ *if the A_i's commute;*
(ii) $P_t(\omega_\sigma; \mathbb{A}_\sigma) = P_t(\omega; \mathbb{A})$ *for any permutation σ;*
(iii) $P_t(\omega; M\mathbb{A}M^*) = MP_t(\omega; \mathbb{A})M^*$ *for any invertible M;*
(iv) $P_t(\omega; \mathbb{A}^{-1})^{-1} = P_{-t}(\omega; \mathbb{A})$*;*
(v) $\sum_{i=1}^n \omega_i A_i \le P_t(\omega; \mathbb{A})$ *for $t \in (1, 2)$;*
(vi) $P_t(\omega; \mathbb{A}) \le (\sum_{i=1}^n \omega_i A_i^{-1})^{-1}$ *for $t \in (-2, -1)$;*
(vii) *If $m \le A_i \le M$ for $i = 1, \ldots, n$ and some scalars $0 < m \le M$, then $m \le P_t(\omega; \mathbb{A}) \le m^{1-t}M^t$ for $t \in (1, 2)$ and $m^{-t}M^{1+t} \le P_t(\omega; \mathbb{A}) \le M$ for $t \in (-2, -1)$;*
(viii) *For $t \in (1, 2)$, $P_t(\omega; A_1, \ldots, A_{n-1}, X) = X$ if and only if $X = P_t(\hat{\omega}; A_1, \ldots, A_{n-1})$.*

Proof. Proofs from (i) to (iv) are similar to those of [13].
(v): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (1, 2)$. Since $\alpha T + (1-\alpha)I \le T^\alpha$ for all $\alpha > 1$ and positive invertible T, we have $(1-t)A + tB \le A\natural_t B$ for $1 < t < 2$ putting $T = A^{-1/2}BA^{-1/2}$ and multiplying $A^{1/2}$ on both sides, see [9, pp.123]. Therefore it follows that

$$X = \sum_{i=1}^n \omega_i (X\natural_t A_i) \ge \sum_{i=1}^n \omega_i((1-t)X + tA_i) = (1-t)X + t\sum_{i=1}^n \omega_i A_i$$

and hence $X \ge \sum_{i=1}^n \omega_i A_i$.
(vi): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (-2, -1)$. Since $X = \left(\sum_{i=1}^n \omega_i (X^{-1}\natural_{-t} A_i^{-1})\right)^{-1}$, it follows that

$$X^{-1} = \sum_{i=1}^n \omega_i (X^{-1}\natural_{-t} A_i^{-1}) \ge \sum_{i=1}^n \omega_i((1+t)X^{-1} + (-t)A_i^{-1}) = (1+t)X^{-1} - t\sum_{i=1}^n \omega_i A_i^{-1}$$

and hence $X \le (\sum_{i=1}^n \omega_i A_i^{-1})^{-1}$ for $t \in (-2, -1)$.
(vii): Put $X = P_t(\omega; \mathbb{A})$ for $t \in (1, 2)$, and $X \ge \sum_{i=1}^n \omega_i A_i \ge m$. Hence we have

$$X = \sum_{i=1}^n \omega_i (X\natural_t A_i) \le \sum_{i=1}^n \omega_i (m\natural_t A_i) = \sum_{i=1}^n \omega_i (m^{1-t}A_i^t) \le m^{1-t}M^t$$

by (i) of Lemma 2.1. Similarly, put $X = P_t(\omega; \mathbb{A})$ for $t \in (-2, -1)$, and $X \le \left(\sum_{i=1}^n \omega_i A_i^{-1}\right)^{-1} \le M$. Hence we have

$$X^{-1} = \sum_{i=1}^n \omega_i (X^{-1}\natural_{-t} A_i^{-1}) \le \sum_{i=1}^n \omega_i (M^{-1}\natural_{-t} A_i^{-1}) = \sum_{i=1}^n \omega_i (M^{-1-t}A_i^t) \le M^{-1-t}m^t$$

and $m^{-t}M^{1+t} \le X$.
(viii): For $1 < t < 2$, $P_t(\omega; A_1, \ldots, A_{n-1}, X) = X \iff X = \sum_{i=1}^{n-1} \omega_i (X\natural_t A_i) + \omega_n X \iff X = \sum_{i=1}^{n-1} \frac{\omega_i}{1-\omega_n}(X\natural_t A_i) \iff X = P_t(\hat{\omega}; A_1, \ldots, A_{n-1})$. □

Theorem 4.6. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ such that $0 < m \le A_i \le M$ for some scalars $0 < m \le M$ and a weight vector $\omega = (\omega_1, \ldots, \omega_n)$. Let $1 < t \le s < 2$. Then*

$$d(P_t(\omega; \mathbb{A}), P_s(\omega; \mathbb{A})) \le \frac{s-t}{(2-s)(2-t)}\left[t\Delta(\mathbb{A}) + \log K\left(m/M, (M/m)^t, t\right)\right],$$

where the generalized Kantorovich constant $K(m, M, t)$ is defined by (1) and $\Delta(\mathbb{A}) = \max_{1\le i,j\le n}\{d(A_i, A_j)\}$ denotes the d-diameter of $\mathbb{A} = (A_1, \ldots, A_n)$.

Proof. Put $X = P_t(\omega; \mathbb{A})$ and $Y = P_s(\omega; \mathbb{A})$. Since $m \le A_i \le M$ for $i = 1, \dots, n$ and $m \le X \le m^{1-t}M^t$, we have $m/MA_i \le X \le (M/m)^t A_i$ and $m/MA_i \le A_j \le M/mA_i$ for $i, j = 1, \dots, n$. It follows from [18, Proposition 4] that $K(m/M, M/m, t) \le K(m/M, (M/m)^t, t)$ for $1 < t < 2$. By Theorem 3.3, it follows that

$$\begin{aligned} d(X, A_j) &= d(\sum_{i=1}^n \omega_i (X \natural_t A_i), \sum_{i=1}^n \omega_i A_j) \\ &\le \max_{1 \le i \le n} \{d(X \natural_t A_i, A_j \natural_t A_j)\} \\ &\le \max_{1 \le i \le n} \left\{(t-1)d(X, A_j) + td(A_i, A_j) + \log K(m/M, (M/m)^t, t)\right\} \\ &= (t-1)d(X, A_j) + t\Delta(\mathbb{A}) + \log K(m/M, (M/m)^t, t) \end{aligned}$$

for $j = 1, \dots, n$ and hence we have

$$d(X, A_j) \le \frac{t}{2-t}\Delta(\mathbb{A}) + \frac{1}{2-t}\log K(m/M, (M/m)^t, t).$$

By Lemma 4.1, we have

$$\begin{aligned} d(X, Y) = d(Y, X) &= d(\sum_{i=1}^n \omega_i (Y \natural_s A_i), \sum_{i=1}^n \omega_i (X \natural_t A_i)) \\ &\le \max_{1 \le i \le n} \{d(Y \natural_s A_i, X \natural_t A_i)\} \\ &\le \max_{1 \le i \le n} \{d(Y \natural_s A_i, X \natural_s A_i) + d(X \natural_s A_i, X \natural_t A_i)\} \\ &\le \max_{1 \le i \le n} \{(s-1)d(Y, X) + (s-t)d(X, A_i)\} \\ &\le (s-1)d(X, Y) + (s-t)\left[\frac{t}{2-t}\Delta(\mathbb{A}) + \frac{1}{2-t}\log K\left(m/M, (M/m)^t, t\right)\right] \end{aligned}$$

and hence we have $d(X, Y) \le \frac{s-t}{2-s}\left[\frac{t}{2-t}\Delta(\mathbb{A}) + \frac{1}{2-t}\log K\left(m/M, (M/m)^t, t\right)\right]$. ◻

Theorem 4.7. *Let $\mathbb{A} = (A_1, \dots, A_n)$ and $\mathbb{B} = (B_1, \dots, B_n)$ such that $0 < m_1 \le A_i \le M_1$ and $0 < m_2 \le B_i \le M_2$ for $i = 1, \dots, n$ for some scalars $0 < m_1 \le M_1$ and $0 < m_2 \le M_2$. Then for each $1 < t < 2$*

$$d(P_t(\omega; \mathbb{A}), P_t(\omega; \mathbb{B})) \le \frac{t}{2-t}\max_{1 \le i \le n}\{d(A_i, B_i)\} + \frac{1}{2-t}\log K_1(t),$$

where $K_1(t) = \max\{K(m_2/m_1^{1-t}M_1^t, m_2^{1-t}M_2^t/m_1, t), K(m_2/M_1, M_2/m_1, t)\}$.

Proof. For fixed $t \in (1, 2)$, put $X = P_t(\omega; \mathbb{A})$ and $Y = P_t(\omega; \mathbb{B})$. Since $m_1 \le X \le m_1^{1-t}M_1^t$ and $m_2 \le Y \le m_2^{1-t}M_2^t$, we have $m_2/m_1^{1-t}M_1^t X \le Y \le m_2^{1-t}M_2^t/m_1 X$ and $m_2/M_1 A_i \le$

$B_i \leq M_2/m_1 A_i$ for $i = 1, \ldots, n$. Then it follows from Theorem 3.3 that

$$\begin{aligned} d(X,Y) &= d(\sum_{i=1}^n \omega_i (X \natural_t A_i), \sum_{i=1}^n \omega_i (Y \natural_t B_i)) \\ &\leq \max_{1\leq i\leq n} \{d(X\natural_t A_i, Y\natural_t B_i)\} \\ &\leq \max_{1\leq i\leq n} \{(t-1)d(X,Y) + td(A_i,B_i) + \log K_1(t)\} \\ &= (t-1)d(X,Y) + t \max_{1\leq i\leq n} \{d(A_i,B_i)\} + \log K_1(t). \end{aligned}$$

□

Remark 4.8. *As $t \to 1$ in Theorem 4.7, we have $K_1(t) \to 1$. Since $P_1(\omega;\mathbb{A}) = \sum_{i=1}^n \omega_i A_i$ and $P_1(\omega;\mathbb{B}) = \sum_{i=1}^n \omega_i B_i$, this corresponds to $d(\sum_{i=1}^n \omega_i A_i, \sum_{i=1}^n \omega_i B_i) \leq \max_{1\leq i\leq n}\{d(A_i,B_i)\}$, see* [12, Lemma 2.4].

5. Monotonicity of power means

In this section, we consider the monotonicity of $P_t(\omega;\mathbb{A})$ for $1 < t < 2$. Before this, in the case of $n = 2$, we consider an explicit form of $P_t(1-\alpha,\alpha;A,B)$ for $1 < t < 2$ and moreover a positive solution of

$$X = (1-\alpha)(X\natural_t A) + \alpha(X\natural_t B) \qquad \text{for } t \in \mathbb{R}. \tag{2}$$

The solution of (2) is $X = A m_{t,\alpha} B = A^{1/2}\left((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^t\right)^{1/t} A^{1/2}$. Indeed, put $C = A^{-1/2}BA^{-1/2}$ and $Y = ((1-\alpha)I + \alpha C^t)^{1/t}$. Then it follows that

$$\begin{aligned} (1-\alpha)(X\natural_t A) + \alpha(X\natural_t B) &= A^{1/2}\left((1-\alpha)(Y\natural_t I) + \alpha(Y\natural_t C)\right) A^{1/2} \\ &= A^{1/2}\left((1-\alpha)Y^{1-t} + \alpha Y^{1-t}C^t\right) A^{1/2} \\ &= A^{1/2}\left(Y^{1-t}((1-\alpha)I + \alpha C^t)\right) A^{1/2} \\ &= A^{1/2} Y A^{1/2} = X. \end{aligned}$$

It is known that the solution $X = A m_{t,\alpha} B$ for $\alpha \in [0,1]$ is nondecreasing for $t \in \mathbb{R}$, see [11, Theorem 5.21]. However, in the case of $n \geq 3$, we have the following order relation:

Theorem 5.1. *Let $\mathbb{A} = (A_1,\ldots,A_n) \in \mathbb{P}^n$ such that $m \leq A_i \leq M$ for $i = 1,\ldots,n$ and some scalars $0 < m \leq M$. Then for each $1 < t \leq s < 2$*

$$(m/M)^{\frac{t(s-1)}{2-s}} P_s(\omega;\mathbb{A}) \leq P_t(\omega;\mathbb{A}) \leq (M/m)^{\frac{s(t-1)}{2-t}} P_s(\omega;\mathbb{A}). \tag{3}$$

To prove Theorem 5.1, we need the following two lemmas. The following lemma is the complement of the Löwner-Heinz inequality:

Lemma 5.2. ([14, Lemma 2.2]) *If $Y \leq X$, then $X^p \leq \| X^{p/2}Y^{-p}X^{p/2} \| \, Y^p$ for $0 < p < 1$.*

Lemma 5.3. *For $1 < t < 2$, define $f(X) = \sum_{i=1}^n \omega_i(X\natural_t A_i)$ where A_i are positive invertible operators and ω is a weight vector. Let X, Y be positive invertible operators such that $m_1 \leq Y \leq M_1$ and $m_2 \leq X \leq M_2$ for some scalars $0 < m_i \leq M_i$ $(i = 1,2)$. Then*

$$Y \leq X \quad \Longrightarrow \quad f(X) \leq f(Y) \leq (M_2/m_1)^{t-1} f(X).$$

In particular, f^2 is monotone, that is, if $Y \leq X$, then $f^2(Y) \leq f^2(X)$.

Proof. The assumption $Y \le X$ implies $f(Y) = \sum_{i=1}^n (Y \natural_t A_i) \ge \sum_{i=1}^n (X \natural_t A_i) = f(X)$ by (i) of Lemma 2.1. Put $X_i = A_i^{1/2} X^{-1} A_i^{1/2}, Y_i = A_i^{1/2} Y^{-1} A_i^{1/2}$ for $i = 1, \ldots, n$. Since $X^{-1} \le Y^{-1}$, we have $X_i \le Y_i$ and it follows from Lemma 5.2 and $0 < t-1 < 1$ that

$$Y_i^{t-1} \le \| Y_i^{(t-1)/2} X_i^{-(t-1)} Y_i^{(t-1)/2} \| X_i^{t-1}$$

for $i = 1, \ldots, n$ and hence

$$Y \natural_t A_i \le \| Y_i^{(t-1)/2} X_i^{-(t-1)} Y_i^{(t-1)/2} \| X \natural_t A_i.$$

Also, by the Araki-Cordes inequality [11, pp.67], we have

$$\begin{aligned} &\| Y_i^{(t-1)/2} X_i^{-(t-1)} Y_i^{(t-1)/2} \| \le \| Y_i^{1/2} X_i^{-1} Y_i^{1/2} \|^{t-1} \\ &= r((A_i^{1/2} Y^{-1} A_i^{1/2})(A_i^{-1/2} X A_i^{-1/2}))^{t-1} \\ &= r(XY^{-1})^{t-1} = \| X^{1/2} Y^{-1} X^{1/2} \|^{t-1} \le (M_2/m_1)^{t-1}. \end{aligned}$$

Therefore, we have

$$f(Y) = \sum_i^n \omega_i (Y \natural_t A_i) \le (M_2/m_1)^{t-1} \sum_i^n \omega_i (X \natural_t A_i) = (M_2/m_1)^{t-1} f(X).$$

□

Proof of Theorem 5.1 Put $f(X) = \sum_{i=1}^n \omega_i (X \natural_t A_i)$ and then $P_t(\omega; \mathbb{A}) = \lim_{k \to \infty} f^k(X)$ for any $X \in \mathbb{P}$. Since $X \natural_t A_i = X \natural_{t/s} (X \natural_s A_i)$, it follows from $0 < t/s \le 1$ that

$$\begin{aligned} f(X) &= \sum_{i=1}^n \omega_i (X \natural_t A_i) = \sum_{i=1}^n \omega_i \left[X \sharp_{t/s} (X \natural_s A_i) \right] \\ &\le \sum_{i=1}^n \omega_i \left[(1 - \frac{t}{s}) X + \frac{t}{s} (X \natural_s A_i) \right] \\ &= (1 - \frac{t}{s}) X + \frac{t}{s} \sum_{i=1}^n \omega_i (X \natural_s A_i). \end{aligned}$$

If we put $X_0 = P_s(\omega; \mathbb{A})$, then we have

$$f(X_0) \le (1 - \frac{t}{s}) X_0 + \frac{t}{s} \sum_{i=1}^n \omega_i (X_0 \natural_s A_i) = (1 - \frac{t}{s}) X_0 + \frac{t}{s} X_0 = X_0.$$

Moreover, we have $m \le X_0 \le m^{1-s} M^s$ and $(m^{1-s} M^s)^{1-t} m^t \le f(X_0) \le m^{1-t} M^t$. By Lemma 5.3, it follows that

$$f^2(X_0) \le \left(\frac{m^{1-s} M^s}{(m^{1-s} M^s)^{1-t} m^t} \right)^{t-1} f(X_0) \le (M/m)^{st(t-1)} X_0.$$

Since f^2 is monotonic, we have

$$\begin{aligned} f^4(X_0) &\le f^2 \left((M/m)^{st(t-1)} X_0 \right) = (M/m)^{st(t-1)(1-t)^2} f^2(X_0) \\ &\le (M/m)^{st(t-1)(1-t)^2 + st(t-1)} X_0. \end{aligned}$$

Inductively we have

$$f^{2k}(X_0) \le \left[(M/m)^{st(t-1)} \right]^{\frac{1-(1-t)^{2k}}{1-(1-t)^2}} X_0.$$

As $k \to \infty$, we have the desired inequality $P_t(\omega;\mathbb{A}) \le (M/m)^{\frac{s(t-1)}{2-t}} P_s(\omega;\mathbb{A})$.

Similarly, if we put $Y_0 = P_s(\omega;\mathbb{A})$ and $g(X) = \sum_{i=1}^n \omega_i(X \natural_s A_i)$, then we have the LHS of (3) in Theorem 5.1. □

6. Positive linear maps

In this section, we consider an information monotonicity of $P_t(\omega;\mathbb{A})$ for $1 < t < 2$. Though the Ando inequality $\Phi(A\sharp_t B) \le \Phi(A)\sharp_t\Phi(B)$ holds for any positive linear map and $t \in [0,1]$ (see [1]), the reverse holds in the case of $t \in (1,2)$:

Lemma 6.1. *Let Φ be a positive linear map on $B(H)$ and $A, B \in \mathbb{P}$ such that $mA \le B \le MA$ for some scalars $0 < m \le M$. Then*

$$\Phi(A\natural_t B) \ge \Phi(A)\natural_t\Phi(B) \ge K(m,M,t)^{-1}\Phi(A\natural_t B) \qquad \text{for } 1 < t < 2,$$

where the generalized Kantorovich constant $K(m,M,t)$ is defined by (1).

Proof. We may assume that Φ is strictly positive. For $A, B > 0$, put

$$\Psi(X) = \Phi(A)^{-1/2}\Phi(A^{1/2}XA^{1/2})\Phi(A)^{-1/2}$$

and hence Ψ is a unital positive linear map. By the Jensen inequality [11, Corollary 1.22, Corollary 2.12]

$$K(m,M,t)\Psi(X)^t \ge \Psi(X^t) \ge \Psi(X)^t \qquad \text{for } 1 < t < 2. \tag{4}$$

Therefore, it follows that

$$\begin{aligned}\Phi(A)^{-1/2}\Phi(A\natural_t B)\Phi(A)^{-1/2} &= \Phi(A)^{-1/2}\Phi(A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2})\Phi(A)^{-1/2}\\ &= \Psi((A^{-1/2}BA^{-1/2})^t)\\ &\ge \Psi(A^{-1/2}BA^{-1/2})^t\\ &= \left(\Phi(A)^{-1/2}\Phi(B)\Phi(A)^{-1/2}\right)^t\end{aligned}$$

and we have $\Phi(A\natural_t B) \ge \Phi(A)\natural_t\Phi(B)$. Similarly, by using (4), we have the RHS of Lemma 6.1. □

The operator power means $P_t(\omega;\mathbb{A})$ for $t \in (0,1]$ have an information monotonicity, see [12, Proposition 3.6]. In the case of $t \in (1,2)$, we have the following slightly modification:

Theorem 6.2. *Let $\mathbb{A} = (A_1,\dots,A_n)$ and a weight vector ω. Let Φ be a unital positive linear map. For each $1 < t < 2$*

$$K\left((m/M)^t, M/m, t\right)^{\frac{-1}{t(2-t)}} (m/M)^{\frac{t-1}{2-t}} \Phi(P_t(\omega;\mathbb{A})) \le P_t(\omega;\Phi(\mathbb{A})) \le (M/m)^{\frac{t(t-1)}{2-t}} \Phi(P_t(\omega;\mathbb{A})),$$

where $\Phi(\mathbb{A}) = (\Phi(A_1),\dots,\Phi(A_n))$ and the generalized Kantorovich constant $K(m,M,t)$ is defined by (1).

Proof. Put $X = P_t(\omega;\mathbb{A})$ and $f(Y) = \sum_{i=1}^n \omega_i(Y\natural_t\Phi(A_i))$ for any $Y \in \mathbb{P}$. Since Φ is unital, we have $m \le \Phi(X) \le m^{1-t}M^t$. Since $f(\Phi(X)) = \sum_{i=1}^n \omega_i(\Phi(X)\natural_t\Phi(A_i))$, we have $(m^{1-t}M^t)^{1-t}m^t \le f(\Phi(X)) \le m^{1-t}M^t$. It follows from $f(\Phi(X)) \le \Phi(X)$ that $f^2(\Phi(X)) \le (M/m)^{t^2(t-1)}f(\Phi(X))$. Hence we have

$$f^3(\Phi(X)) \le f^2(\Phi(X)) \le (M/m)^{t^2(t-1)}\Phi(X).$$

Inductively it follows that

$$f^{2k-1}(\Phi(X)) \le \left[(M/m)^{t^2(t-1)}\right]^{\frac{1-(1-t)^{2k}}{1-(1-t)^2}} \Phi(X)$$

and as $k \to \infty$ we have

$$P_t(\omega;\Phi(\mathbb{A})) \le \left[(M/m)^{t^2(t-1)}\right]^{\frac{1}{1-(1-t)^2}} \Phi(P_t(\omega;\mathbb{A})) = (M/m)^{\frac{t(t-1)}{2-t}} \Phi(P_t(\omega;\mathbb{A})).$$

Conversely, since $K((m/M)^t, M/m, t)^{-1}\Phi(X) \le f(\Phi(X))$, we have

$$K((m/M)^t, M/m, t)^{-1}(M/m)^{t(t-1)}\Phi(X) \le f^2(\Phi(X)).$$

Since f^2 is monotone, it follows that

$$\begin{aligned} f^3(\Phi(X)) &\ge f^2(K((m/M)^t, M/m, t)^{-1}\Phi(X)) = K((m/M)^t, M/m, t)^{-(1-t)^2} f^2(\Phi(X)) \\ &\ge K((m/M)^t, M/m, t)^{-1-(1-t)^2}(m/M)^{t(t-1)}\Phi(X). \end{aligned}$$

Inductively, we have

$$K((m/M)^t, M/m, t)^{-\frac{1-(1-t)^{2(k-1)}}{1-(1-t)^2}} \left[(m/M)^{t(t-1)}\right]^{\frac{1}{1-(1-t)^2}} \Phi(X) \le f^{2k-1}(\Phi(X))$$

for $k = 1, 2 \cdots$ and as $k \to \infty$

$$K((m/M)^t, M/m, t)^{\frac{-1}{t(2-t)}}(m/M)^{\frac{t-1}{2-t}}\Phi(P_t(\omega;\mathbb{A}) \le P_t(\omega;\Phi(\mathbb{A})).$$

□

Remark 6.3. As $t \to 1$ in Theorem 6.2, we have $K((m/M)^t, M/m, t)^{\frac{-1}{t(2-t)}}(m/M)^{\frac{t-1}{2-t}} \to 1$ and $(M/m)^{t^2(t-1)} \to 1$. This corresponds to $P_1(\omega;\Phi(\mathbb{A})) = \sum_{i=1}^n \omega_i\Phi(A_i) = \Phi(\sum_{i=1}^n \omega_i A_i) = \Phi(P_1(\omega;\mathbb{A}))$.

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