

On a construction of multiwavelets

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ABSTRACT. A class of r -regular multiwavelets is introduced with the appropriate notation and definitions. The oscillation properties of the multiwavelet orthonormal systems are obtained in Lemma 1 and Corollary 1 without assuming any oscillation property on the part of the scaling functions. The existence of r -regular multiwavelets in $L^2(\mathbb{R}^n)$ is established in Theorem 1. In Theorem 2, a particular r -regular multiresolution analysis for multiwavelets is obtained from an r -regular multiresolution analysis for uniwavelets. In Theorem 3, an r -regular multiresolution analysis of *split-type* multiwavelets, which are perhaps the simplest multiwavelets, is easily obtained by using an r -regular multiresolution analysis for uniwavelets and a $(2^n - 1)$ -fold regular multiresolution analysis for uniwavelets. For some split-type multiwavelets, the support or width of the wavelets is shorter than the support or width of the scaling functions without loss of regularity nor of vanishing moments. Examples of split-type multiwavelets in $L^2(\mathbb{R})$ are constructed and illustrated by means of figures. Symmetry and antisymmetry are preserved in the case of infinite support.

Keywords and phrase. multiwavelets, multiwavelets of split type, split wavelets, regular multiresolution, vanishing moments

RÉSUMÉ. On introduit une classe de multi-ondelettes r -régulières. Par le lemme 1 et le corollaire 1 on déduit le caractère oscillant des systèmes orthonormaux de multi-ondelettes sans supposer aucun caractère oscillant sur les fonctions d'échelle. Au théorème 1, on démontre l'existence de multi-ondelettes de $L^2(\mathbb{R}^n)$ r -régulières. Au théorème 2, on obtient une analyse multirésolution r -régulière particulière pour multi-ondelettes à partir d'une analyse multirésolution r -régulière pour uni-ondelettes. Au théorème 3, on obtient facilement une analyse multirésolution r -régulière pour multi-ondelettes du type “séparé”, qui sont peut-être les multi-ondelettes les plus simples, au moyen d'une analyse multirésolution r -régulière pour uni-ondelettes et $(2^n - 1)$ analyses multirésolution régulières pour uni-ondelettes. Pour certaines multi-ondelettes du type “séparé”, le support ou l'écart des ondelettes est plus faible que le support ou l'écart des fonctions d'échelle sans perte de régularité ni de caractère oscillant. On construit des multi-ondelettes dans $L^2(\mathbb{R}^n)$ du type “séparé” dont on illustre le tracé dans des figures. Cette construction préserve la symétrie et l'antisymétrie dans le cas de support infini.

1. INTRODUCTION

Since wavelets are solutions of multiscale equations, they can hardly be studied as mathematical objects and in the applications without the use of computers. This paper is no exception.

Multiwavelets consist of several scaling functions and wavelets. It is believed that multiwavelets are ideally suited to multichannel signals like color images which are two-dimensional three-channel signals and stereo audio signals which are one-dimensional two-channel signals.

For instance, a two-channel signal consists of a two-vector sequence $\{x_k\}$ of bits. The low-pass and high-pass filters are 2×2 function matrices corresponding to a 2-scaling function and a 2-wavelet, respectively.

Multiscaling functions and multiwavelets can simultaneously have orthogonality, linear phase, symmetry and compact support. This situation cannot occur with real uniscaling functions and real uniwavelets.

The simplest uniwavelet in $L^2(\mathbb{R}^1)$ is the Haar system [1, Section 3.2] with the indicator function of the interval $[0, 1]$ as scaling function (see Fig. 1 in Section 5). Alpert [2] generalized the Haar system to one-dimensional discontinuous multiwavelets with vanishing moments in $L^2(\mathbb{R}^1)$.

Using fractal interpolation, Geronimo, Hardin, and Massopust [3] constructed a real-valued one-dimensional symmetric two-scaling function with short support, and Donavan, Geronimo, Hardin, and Massopust [4] constructed a corresponding real-valued one-dimensional two-wavelet (DGHM) with short support. Strang and Strela used matrix methods in the time domain to construct the DGHM two-wavelet in [5], and a nonsymmetric pair in [6].

Assuming sufficiently many vanishing moments of the scaling functions, Ashino and Kametani [7] introduced r -regular multiwavelets in $L^2(\mathbb{R}^n)$ and proved a general existence theorem, following Meyer's general existence theorem (see, in [1], Theorem 2 of Section 3.6 and Proposition 4 of Section 3.7). Jia and Shen [8] investigated multiresolution on the basis of shift-invariant spaces, proved a general existence theorem and gave examples to illustrate the general theory. Using results of Lawton [9] on complex-valued filters, Cooklev [10] and Cooklev *et al.* [11] obtained one-dimensional perfect-reconstruction two-filter banks given by a pair of analysing and synthesizing orthogonal linear-phase two-channel multiwavelet filters. Very recently, Plonka [12], Cohen, Daubechies and Plonka [13], Plonka and Strela [14], Shen [15], Strela [16], and many others, have obtained important results on the existence, regularity, orthogonality and symmetry of multiwavelets.

Definitions of filters and filter banks can be found, for instance, in [17], [18] and [19].

In the first part of this paper, general existence theorems are given. In the second part, split-type multiwavelets are constructed from existing uniwavelets. These are perhaps the simplest multiwavelets.

Section 2 contains the notation, definitions and Lemma 1. Here, the definition of r -regularity, which is based on the regularity of the multiwavelet functions, differs from the definition of r -regularity, based on the number of vanishing moments of

the multiscaling function, used by Ashino and Kametani [7]. A 0-regular multiresolution analysis for uniwavelets, as defined by Meyer [1], will be simply called *regular* multiresolution analysis for uniwavelets. Daubechies' Theorem 5.5.1 on vanishing moments for biorthogonal systems and Corollary 5.5.2 for orthonormal systems in [17, pp. 153–154] are generalized to the multidimensional cases in Lemma 1 and Corollary 1, respectively. Corollary 1 is used to define a multiresolution analysis for r -regular multiwavelets without assuming that the scaling functions have any vanishing moments.

In Section 3, given a multiresolution analysis as defined in Definition 5, multidimensional r -regular multiwavelets can be constructed by the general existence Theorem 1. A particular construction of multiwavelets is obtained in Theorem 2 by constructing an r -regular multiresolution analysis for multiwavelets from an r -regular multiresolution analysis for uniwavelets.

In Section 4, Theorem 3 gives an easy construction of an r -regular multiresolution analysis for *split-type* multiwavelets obtained by means of an r -regular multiresolution analysis for uniwavelets and a $(2^n - 1)$ -fold regular multiresolution analysis for uniwavelets. For some split-type multiwavelets, the support or width of the wavelets is shorter than the support or width of the scaling functions without loss of regularity or vanishing moments.

In Section 5, we study one-dimensional split-type multiwavelets which are shown to be identical to the *split wavelets* defined in Daubechies [17, Section 10.5]. Thus, multidimensional r -regular split-type multiwavelets constructed in Theorem 3 are a natural higher-dimensional generalization of split wavelets. Finally, we give examples and figures of one-dimensional multiwavelets of split type. Symmetry and antisymmetry are preserved in the case of infinite support.

2. NOTATION, DEFINITIONS AND A LEMMA

In this section, we introduce the notation and definitions of multidimensional multiwavelets, and prove Lemma 1 on vanishing moments of biorthogonal functions.

Notation 1. The following notation will be used.

- $f_{jk}(x)$ is a scaled and shifted function,

$$f_{jk}(x) = 2^{nj/2} f(2^j x - k), \quad j \in \mathbb{Z}, k \in \mathbb{Z}^n, \quad f \in L^2(\mathbb{R}^n). \quad (2.1)$$

- F_{jk} is a d -vector of scaled and shifted functions,

$$F_{jk} = ((f_1)_{jk}, \dots, (f_d)_{jk}), \quad j \in \mathbb{Z}, k \in \mathbb{Z}^n, \quad F = (f_1, \dots, f_d) \in L^2(\mathbb{R}^n)^d.$$

- $R = \{0, 1\}^n$ is the set of 2^n vertices of the n -dimensional unit cube.
- $E = R \setminus \{(0, \dots, 0)\}$ is the set of vertices of R less the origin.
- $D = \{1, \dots, d\}$ for a positive integer d .
- $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, including zero.
- $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \cong [0, 2\pi[$ is the one-dimensional torus.
- $2\mathbb{T} = \mathbb{R}/(\pi\mathbb{Z}) \cong [0, \pi[$.

- $r \in \mathbb{N}$ throughout the paper.
- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \in \mathbb{N}$, is a multi-index of nonnegative integers.
- $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ is the length of the multi-index α .
- $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$.
- $m(\xi)$, with $\xi \in \mathbb{R}^n$, is $2\pi\mathbb{Z}^n$ -periodic if it is 2π -periodic in each ξ_j , $j = 1, 2, \dots, n$, that is, $m(\xi)$ is a function defined on the n -dimensional torus \mathbb{T}^n .
- $U(n)$, $n \in \mathbb{N} \setminus \{0\}$, is the unitary group of order n , that is, the group of $n \times n$ unitary matrices.
- The superscript T for “transpose” is used to denote column vectors, like in $(u_1, \dots, u_s)^T$ and $(u_\sigma)_{\sigma \in \Sigma}^T$.
- $L^2(\mathbb{R}^n)^d := (L^2(\mathbb{R}^n), \dots, L^2(\mathbb{R}^n))^T$ and $L^2(\mathbb{T}^n)^d := (L^2(\mathbb{T}^n), \dots, L^2(\mathbb{T}^n))^T$. ■

A function $f(x)$ and its Fourier transform $\hat{f}(\xi)$ are related by the formulae

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{f}(\xi) d\xi.$$

Let

$$\mathcal{B}^r(\mathbb{R}^n) := \left\{ f(x) \in C^r(\mathbb{R}^n); \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq r}} |\partial^\alpha f(x)| < \infty \right\},$$

and denote by $\mathcal{R}_r(\mathbb{R}^n)$ the set of functions f in $L^1(\mathbb{R}^n)$ which satisfy the following two conditions:

$$f(x) \in \mathcal{B}^r(\mathbb{R}^n), \tag{2.2}$$

$$x^\alpha f(x) \in L^1(\mathbb{R}^n), \quad |\alpha| \leq r. \tag{2.3}$$

Denote by $\mathcal{S}_r(\mathbb{R}^n)$ the vector space of functions $f \in L^\infty(\mathbb{R}^n)$ which satisfy the following two conditions:

$$f^{(\alpha)}(x) := \partial_x^\alpha f(x) \in L^\infty(\mathbb{R}^n), \quad |\alpha| \leq r, \tag{2.4}$$

and, for every positive number N , there exists a positive number C_N such that

$$|f^{(\alpha)}(x)| \leq C_N (1 + |x|)^{-N}, \quad \text{a.a. } x, \quad |\alpha| \leq r. \tag{2.5}$$

Since $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, it follows that $\mathcal{R}_r(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for $r \in \mathbb{N}$. The space $\mathcal{S}_r(\mathbb{R}^n)$ was introduced by Y. Meyer [1, Section 2.6].

Definition 1. A family $\{\Psi_\varepsilon\}_{\varepsilon \in E}$ is called a $2^n - 1$ family of *multiwavelets*,

$$\Psi_\varepsilon := (\psi_{\varepsilon 1}, \dots, \psi_{\varepsilon d}) \in L^2(\mathbb{R}^n)^d,$$

if

$$\{(\psi_{\varepsilon \delta})_{jk}(x) := 2^{nj/2} \psi_{\varepsilon \delta}(2^j x - k)\}_{\varepsilon \in E, \delta \in D, j \in \mathbb{Z}, k \in \mathbb{Z}^n}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. The functions $(\psi_{\varepsilon \delta})_{jk}$ are called *multiwavelets*.

Remark 1. An intuitive geometric explanation why $2^n - 1$ multiwavelets are needed is as follows. If, after approximating \mathbb{R}^n by the lattice \mathbb{Z}^n , we want to approximate it by the more refined lattice $\frac{1}{2}\mathbb{Z}^n$, then we need to add $2^n - 1$ extra points for every point in \mathbb{Z}^n and hence $2^n - 1$ extra approximating functions in $L^2(\mathbb{R}^n)^d$.

Definition 2. A family of multiwavelets $\{\Psi_\varepsilon\}_{\varepsilon \in E}$ is said to be *regular* if

$$\psi_{\varepsilon\delta} \in \mathcal{S}_0(\mathbb{R}^n), \quad \varepsilon \in E, \quad \delta \in D,$$

and it is said to be *r-regular* if

$$\psi_{\varepsilon\delta} \in \mathcal{R}_r(\mathbb{R}^n) \cap \mathcal{S}_r(\mathbb{R}^n), \quad \varepsilon \in E, \quad \delta \in D.$$

We remark that $\psi_{\varepsilon\delta}$ is *r-regular* in the Meyer sense if $\psi_{\varepsilon\delta} \in \mathcal{S}_r(\mathbb{R}^n)$. Thus the Haar wavelet $H\psi(x)$ is regular (see Fig. 1 in Section 5).

The scalar product of the functions f and g in $L^2(\mathbb{R}^n)$ is denoted by

$$\langle f, g \rangle := \int f(x) \overline{g(x)} dx.$$

The following lemma on the number of vanishing moments of a function $\tilde{f} \in \mathcal{R}^r(\mathbb{R}^n)$ reduces to Theorem 5.5.1 in [17], p. 153, in the one-dimensional case.

Lemma 1. Suppose that f and \tilde{f} are two functions such that

$$\langle f_{jk}, \tilde{f}_{j'k'} \rangle = \delta_{jj'} \delta_{kk'}, \quad (2.6)$$

with $f(x) \in \mathcal{B}^r(\mathbb{R}^n)$, $\widehat{f}(\xi) \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $x^\alpha \tilde{f}(x) \in L^1(\mathbb{R}^n)$ for $|\alpha| \leq r$. Then the following oscillation property is satisfied:

$$\int x^\alpha \tilde{f}(x) dx = 0, \quad \text{for } |\alpha| \leq r. \quad (2.7)$$

Proof. The proof is by induction on the length $|\alpha|$ of the multi-index α . By (2.6), for $j \neq 0$ we have

$$\begin{aligned} 0 &= \langle f, \tilde{f}_{j,2^{j+p}k} \rangle \\ &= 2^{jn/2} \int f(x) \overline{\tilde{f}(2^j x - 2^{j+p}k)} dx \\ &= 2^{-jn/2} \int f(2^{-j}y + 2^p k) \overline{\tilde{f}(y)} dy. \end{aligned} \quad (2.8)$$

Hence, we have

$$\int_{\mathbb{R}^n} f(2^{-j}y + 2^p k) \overline{\tilde{f}(y)} dy = 0. \quad (2.9)$$

I. Suppose $\alpha = 0$. The assumption on f implies that the integration over \mathbb{R}^n in (2.9) can be interchanged with the limit as $j \rightarrow \infty$. Thus we have

$$f(2^p k) \int \overline{\tilde{f}(y)} dy = 0.$$

Since the set $\{2^p k; p \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is dense in \mathbb{R}^n and $f(x)$ is continuous, then

$$f(x) \int \overline{\tilde{f}(y)} dy = 0, \quad \forall x \in \mathbb{R}^n,$$

and since $f(x) \not\equiv 0$ we get

$$\int \tilde{f}(x) dx = 0.$$

II. Now suppose that (2.7) holds for $|\alpha| < \ell \leq r$. We show that it holds for $|\alpha| = \ell$. Applying Taylor's Theorem to (2.9) we have

$$\begin{aligned} 0 &= \int f(2^{-j}y + 2^p k) \overline{\tilde{f}(y)} dy \\ &= \sum_{|\alpha| < \ell} \frac{1}{\alpha!} f^{(\alpha)}(2^p k) (2^{-j})^{|\alpha|} \int y^\alpha \overline{\tilde{f}(y)} dy \\ &\quad + \sum_{|\alpha| = \ell} \frac{\ell}{\alpha!} \int dy \int_0^1 (2^{-j}y)^\alpha (1-\theta)^{\ell-1} f^{(\alpha)}(2^p k + \theta 2^{-j}y) \overline{\tilde{f}(y)} d\theta. \end{aligned}$$

Hence, by the induction hypothesis and after division by $2^{-j\ell}$, this expression reduces to

$$\sum_{|\alpha| = \ell} \frac{\ell}{\alpha!} \int dy \int_0^1 y^\alpha (1-\theta)^{\ell-1} f^{(\alpha)}(2^p k + \theta 2^{-j}y) \overline{\tilde{f}(y)} d\theta = 0.$$

Again, letting $j \rightarrow \infty$, we have

$$\sum_{|\alpha| = \ell} \frac{\ell}{\alpha!} \int dy \int_0^1 y^\alpha (1-\theta)^{\ell-1} f^{(\alpha)}(2^p k) \overline{\tilde{f}(y)} d\theta = 0$$

and hence

$$\sum_{|\alpha| = \ell} \frac{1}{\alpha!} f^{(\alpha)}(2^p k) \int y^\alpha \overline{\tilde{f}(y)} dy = 0.$$

If we put

$$c_\alpha = \frac{1}{\alpha!} \int y^\alpha \overline{\tilde{f}(y)} dy,$$

then we can write

$$\sum_{|\alpha| = \ell} c_\alpha f^{(\alpha)}(2^p k) = 0. \quad (2.10)$$

As before, (2.10) implies that

$$\sum_{|\alpha| = \ell} c_\alpha f^{(\alpha)}(x) = 0, \quad \forall x \in \mathbb{R}^n. \quad (2.11)$$

Taking the Fourier transform of (2.11), we have

$$\sum_{|\alpha|=\ell} c_\alpha \xi^\alpha \hat{f}(\xi) = 0.$$

If there exists α with $|\alpha| = \ell$ such that $c_\alpha \neq 0$, then the Lebesgue measure of the set

$$\left\{ \xi \in \mathbb{R}^n; \sum_{|\alpha|=\ell} c_\alpha \xi^\alpha = 0 \right\}$$

is zero. Hence $\widehat{f}(\xi) \equiv 0$, a.a. ξ . This contradicts the fact that $f \not\equiv 0$ by (2.6). Thus we have $c_\alpha = 0$ for all $|\alpha| = \ell$. This means that

$$\int x^\alpha \tilde{f}(x) dx = 0, \quad |\alpha| = \ell.$$

Therefore, by induction on ℓ , we have

$$\int x^\alpha \tilde{f}(x) dx = 0, \quad |\alpha| \leq r. \quad \square$$

Corollary 1. *Let $\psi \in \mathcal{R}_r(\mathbb{R}^n)$ and suppose that $\{\psi_{jk}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^n}$ is an orthonormal system of $L^2(\mathbb{R}^n)$. Then the following oscillation property is satisfied:*

$$\int x^\alpha \psi(x) dx = 0, \quad \text{for } |\alpha| \leq r. \quad (2.12)$$

Proof. The relations (2.12) follow immediately from Lemma 1, with $\psi = f = \tilde{f}$. \square

Remark 2. By Corollary 1, every r -regular multiwavelet $\psi_{\varepsilon\delta}$ has the oscillation property (2.12), which is equivalent to $\widehat{\psi}_{\varepsilon\delta}^{(\alpha)}(0) = 0$, for $|\alpha| \leq r$.

A central feature of multiwavelets is their localizing property in both the x - and ξ -spaces. Since the support of $(\psi_{\varepsilon\delta})_{jk}$ becomes very big as $j \rightarrow -\infty$, even if every $(\psi_{\varepsilon\delta})_{jk}$ has compact support, we look for appropriate bases for the spaces spanned by $\{(\psi_{\varepsilon\delta})_{jk}\}_{\varepsilon \in E, \delta \in D, j \in \{-1, -2, \dots\}, k \in \mathbb{Z}^n}$ by considering the following subspaces of $L^2(\mathbb{R}^n)$.

Notation 2. For all $j \in \mathbb{Z}$, let

$$\begin{aligned} W_{j\delta} &:= \overline{\text{Span}\{(\psi_{\varepsilon\delta})_{jk}\}_{\varepsilon \in E, k \in \mathbb{Z}^n}}, \quad \delta \in D; \\ V_{j\delta} &:= \bigoplus_{k=-\infty}^{j-1} W_{k\delta}, \quad \delta \in D; \quad W_j := \bigoplus_{\delta \in D} W_{j\delta}; \quad V_j := \bigoplus_{k=-\infty}^{j-1} W_k. \end{aligned} \quad (2.13)$$

Definition 3. A column function vector $\Phi := (\varphi_1, \dots, \varphi_d)^T \in (V_0)^d$ is called a *multiscaling function* if the family of translates $\{(\varphi_\delta)_{0,k}\}_{\delta \in D, k \in \mathbb{Z}^n}$ is an orthonormal basis of V_0 . The multiscaling function $\Phi(x)$ is said to be *regular* if each φ_δ satisfies

$$\varphi_\delta \in \mathcal{S}_0(\mathbb{R}^n), \quad \delta \in D,$$

and it is said to be *r-regular* if each φ_δ satisfies

$$\varphi_\delta \in \mathcal{R}_r(\mathbb{R}^n) \cap \mathcal{S}_r(\mathbb{R}^n), \quad \delta \in D.$$

Remark 3. In [7], it was shown that the condition $\sum_{\delta \in D} |\widehat{\varphi}_\delta(0)|^2 = 1$ is necessary for the existence of an *r-regular* multiscaling function. Hence there exists $\delta \in D$ such that $\int \varphi_\delta(x) dx \neq 0$. In the case of uniwavelets, as stated in [1, Section 2.10, Proposition 7],

$$\int \varphi(x) dx \neq 0 \implies \int x^\alpha \varphi(x) dx = 0 \quad \text{for } 1 \leq |\alpha| \leq 2r + 1,$$

by suitably changing $\varphi(x)$. But, in the case of multiwavelets, this implication is still open.

Definition 4. Series expansions in terms of the orthonormal basis $\{(\psi_{\varepsilon\delta})_{jk}\}_{\substack{\varepsilon \in E, \delta \in D, \\ j \in \mathbb{N}, k \in \mathbb{Z}^n}} \cup \{(\varphi_\delta)_{0,k}\}_{\delta \in D, k \in \mathbb{Z}^n}$ are called *multiwavelet expansions*.

Remark 4. By Definition 4, $\{(\psi_{\varepsilon\delta})_{jk}\}_{\varepsilon \in E, \delta \in D, k \in \mathbb{Z}^n} \cup \{(\varphi_\delta)_{0,k}\}_{\delta \in D, k \in \mathbb{Z}^n}$ is an orthonormal basis of V_{j+1} for every $j \in \mathbb{Z}$.

To construct *r-regular* multiwavelets $\Psi_\varepsilon(x)$, for $\varepsilon \in E$, one uses the multiscaling function $\Phi(x)$ given by a *multiresolution analysis* defined below.

Definition 5. An increasing sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$,

$$\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots,$$

is called a *multiresolution analysis* if it satisfies the following four properties:

- (a) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\cup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$;
- (b) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$;
- (c) $f(x) \in V_0$ if and only if $f(x - k) \in V_0$ for every $k \in \mathbb{Z}^n$;
- (d) there exists a function vector $\Phi(x) := (\varphi_1(x), \dots, \varphi_d(x))^T \in (V_0)^d$ such that $\{\varphi_\delta(x - k)\}_{\delta \in D, k \in \mathbb{Z}^n}$ forms an orthonormal basis of V_0 .

Definition 6. A multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ is said to be *regular* (or *r-regular*) if the multiscaling function $\Phi(x) \in (V_0)^d$ appearing in part (d) of Definition 5 is regular (or *r-regular*).

3. GENERAL EXISTENCE THEOREM FOR MULTIWAVELETS

We have the following general existence theorem.

Theorem 1. *Let an r -regular multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of multiwavelets be given in $L^2(\mathbb{R}^n)$. Then there exists an r -regular family $\{\Psi_\varepsilon\}_{\varepsilon \in E}$ of $2^n - 1$ multiwavelets $\Psi_\varepsilon := (\psi_{\varepsilon 1}, \dots, \psi_{\varepsilon d})^T \in V_1^d$, for $\varepsilon \in E$.*

This proof of Theorem 1 is based on Lemmas 2 and 3 and Proposition 1 below.

Lemma 2. *Let a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ with a multiscaling function Φ be given. Then there exists a $d \times d$ $L^2(\mathbb{T}^n)$ -valued matrix*

$$\begin{aligned} M_0(\xi) &:= \left((m_0)_{(d', d'')}(\xi); d' \downarrow 1, \dots, d, d'' \rightarrow 1, \dots, d \right) \\ &= \left((m_0)_{(d', d'')}(\xi) \right)_{(d', d'') \in D \times D} \in \text{Mat}(d \times d; L^2(\mathbb{T}^n)) \end{aligned}$$

such that

$$\widehat{\Phi}(2\xi) = M_0(\xi)\widehat{\Phi}(\xi). \quad (3.1)$$

Proof. The proof of this lemma is simple and will be omitted.

The following notation will be needed in Lemma 3.

Let $J_n = \{0, 1, \dots, 2^n - 1\}$. Any number $\ell \in J_n$ can be written uniquely, in the base two, in the form

$$\ell = c_{n-1}(\ell)2^{n-1} + c_{n-2}(\ell)2^{n-2} + \dots + c_1(\ell)2^1 + c_0(\ell), \quad (3.2)$$

where each $c_k(\ell)$, $k = 0, \dots, n-1$, is either 0 or 1. Write

$$\alpha_{n,\ell} = (c_{n-1}(\ell), c_{n-2}(\ell), \dots, c_1(\ell), c_0(\ell)), \quad \ell \in J_n. \quad (3.3)$$

Hereafter we let $\{\alpha_{n,\ell}\}_{\ell \in J_n}$ define the ordering of R . We shall write $\Psi_0 := \Phi$ and, for short,

$$\Psi_{\alpha_{n,\ell}} = \Psi_\ell = \left(\psi_{\ell\delta} \right)_{\delta \in D}^T \in L^2(\mathbb{T}^n)^d$$

for $\ell \in J_n \setminus \{0\}$. For

$$M_\ell(\xi) := \left((m_\ell)_{(d', d'')}(\xi) \right)_{(d', d'') \in D \times D} \in \text{Mat}(d \times d; L^2(\mathbb{T}^n))$$

with $\ell \in J_n$, define Ψ_ℓ by

$$\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi)\widehat{\Phi}(\xi). \quad (3.4)$$

Put

$$L_{\ell'\ell''}(2\xi) := \sum_{\eta \in R} M_{\ell'}(\xi + \pi\eta) M_{\ell''}(\xi + \pi\eta)^*, \quad \ell', \ell'' \in J_n, \quad (3.5)$$

and

$$L(\xi) := \left(L_{\ell'\ell''}(\xi) \right)_{(\ell', \ell'') \in J_n \times J_n}. \quad (3.6)$$

Then $L_{\ell'\ell''}(2\xi)$ is $\pi\mathbb{Z}^n$ -periodic; thus

$$L_{\ell'\ell''}(\xi) \in \text{Mat}(d \times d; L^1(\mathbb{T}^n))$$

and

$$L(\xi) \in \text{Mat}(2^n \times 2^n; \text{Mat}(d \times d; L^1(\mathbb{T}^n))) \simeq \text{Mat}(2^n d \times 2^n d; L^1(\mathbb{T}^n)).$$

Lemma 3. *The sequence $\{(\psi_{\ell\delta})_{0k}\}_{\ell \in J_n, \delta \in D, k \in \mathbb{Z}^n}$ is an orthonormal system if and only if*

$$L(\xi) = I_{2^n d}, \quad \text{a.a. } \xi. \quad (3.7)$$

Sketch of the proof of Lemma 3. Put

$$\widetilde{M}_{\ell' d'}(\xi) := ((m_{\ell'})_{(d'', d')}(\xi + \pi\alpha_{n, \ell''}))_{(d'', \ell'') \in D \times J_n} \in \text{Mat}(d \times 2^n; L^2(\mathbb{T}^n)), \quad (3.8)$$

for $\ell' \in J_n$, $d' \in D$, and put

$$\widetilde{M}(\xi) := (\widetilde{M}_{\ell' d'}(\xi))_{(\ell', d') \in J_n \times D} \in \text{Mat}(2^n \times d; \text{Mat}(d \times 2^n; L^2(\mathbb{T}^n))). \quad (3.9)$$

Then we have

$$\widetilde{M}(\xi) \widetilde{M}(\xi)^* = L(2\xi). \quad (3.10)$$

Let

$$M(\xi) := (M_{\ell'}(\xi + \pi\alpha_{n, \ell''}))_{(\ell', \ell'') \in J_n \times J_n}. \quad (3.11)$$

The matrix $\widetilde{M}(\xi)$ is obtained by changing the order of the columns of the matrix $M(\xi)$ from the lexicographically ordered set $J_n \times D$ to the lexicographically ordered set $D \times J_n$. \square

Using Lemma 3, we obtain the following proposition which is the main tool in the proof of Theorem 1.

Proposition 1. *The family of functions $\{\Psi_\ell\}_{\ell \in J_n \setminus \{0\}}$ defined by the relations*

$$\widehat{\Psi}_\ell(2\xi) = M_\ell(\xi) \widehat{\Phi}(\xi), \quad (3.12)$$

is a family of multiwavelets if and only if

$$M(\xi) \in U(2^n d), \quad \text{a.a. } \xi. \quad (3.13)$$

Although the definitions of r -regularity in the present Theorem 1 and in Theorem 3 of [7] are different, many details of the respective proofs are rather similar so that we shall only sketch the proof of Theorem 1.

Outline of the proof of Theorem 1. Given a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$, we have $M_0(\xi)$ by Lemma 2. Since the multiresolution analysis is regular, then

$$M_0(\xi) \in \text{Mat}(d \times d; C^\infty(\mathbb{T}^n)).$$

By using a differential geometric result [7, Theorem 1], we can construct a $d \times d$ matrix $M_\ell(\xi) \in \text{Mat}(d \times d; C^\infty(\mathbb{T}^n))$, $\ell \in J_n$, which satisfies (3.12). Represent the elements of $M_\ell(\xi)$ by the Fourier series

$$M_\ell(\xi) = \left(\sum_{k \in \mathbb{Z}^n} \alpha_{\ell d' d'' k} e^{-ik \cdot \xi} \right)_{(d', d'') \in D \times D}, \quad (3.14)$$

whose coefficients, $\alpha_{\ell d' d'' k}$, are rapidly decreasing as $k \rightarrow \infty$. Then (3.12) implies

$$\begin{aligned} 2^{-n} \Psi_\ell(x/2) &= \left(\sum_{k \in \mathbb{Z}^n} \alpha_{\ell d' d'' k} \right)_{(d', d'') \in D \times D} \Phi(x - k) \\ &= \left(\sum_{k \in \mathbb{Z}^n, d'' \in D} \alpha_{\ell d' d'' k} \varphi_{d''}(x - k) \right)_{d' \downarrow 1, \dots, d}. \end{aligned} \quad (3.15)$$

Differentiating (3.15) under the summation sign, we can show that every Ψ_ℓ , $\ell \in J_n \setminus \{0\}$, has the same regularity (2.2) and localization property (2.5) as Φ . One sees, by (2.5), that condition (2.3) is trivial. Hence $\psi_{\varepsilon \delta} \in \mathcal{R}_r(\mathbb{R}^n) \cap \mathcal{S}_r(\mathbb{R}^n)$, where $\varepsilon \in E$ and $\delta \in D$. \square

In the following theorem, we construct an r -regular multiresolution analysis for multiwavelets from an r -regular multiresolution analysis for uniwavelets and construct multiwavelets by applying Theorem 1.

Theorem 2. *Let an r -regular multiresolution analysis $\{\tilde{V}_j\}_{j \in \mathbb{Z}}$ of uniwavelets in $L^2(\mathbb{R}^n)$ be given. Then there exists an r -regular family $\{\Psi_\varepsilon\}_{\varepsilon \in E}$ of $2^n - 1$ multiwavelets with $d = 2^n$.*

Proof. By Theorem 1, we need only construct a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$ of multiwavelets in $L^2(\mathbb{R}^n)$. Put $d = 2^n$ and identify $D \simeq R$. We have an r -regular uniscaling function $\tilde{\varphi}$ and r -regular uniwavelets $\tilde{\psi}_\varepsilon$, $\varepsilon \in E$. Set $\tilde{\psi}_0 := \tilde{\varphi}$ and take the d -dimensional function vector

$$\Phi = (\varphi_\delta)_{\delta \in D}^T := (\tilde{\psi}_\varepsilon)_{\varepsilon \in E}^T,$$

as an r -regular multiscaling function. Define

$$V_j := \overline{\text{Span}\{(\varphi_\delta)_{jk}\}}_{\delta \in D, k \in \mathbb{Z}^n}, \quad j \in \mathbb{Z}. \quad (3.16)$$

Then

$$\begin{aligned} V_j &= \overline{\text{Span}\{(\tilde{\varphi})_{jk}\}}_{k \in \mathbb{Z}^n} \oplus \overline{\text{Span}\{(\tilde{\psi}_\varepsilon)_{jk}\}}_{\varepsilon \in E, k \in \mathbb{Z}^n} \\ &= \tilde{V}_j \oplus (\bigoplus_{\varepsilon \in E} \tilde{W}_{j\varepsilon}) = \tilde{V}_{j+1}. \end{aligned}$$

Hence $\{V_j\}_{j \in \mathbb{Z}}$ is an increasing sequence of closed subspaces of $L^2(\mathbb{R}^n)$ and properties (a) and (b) of Definition 5 are satisfied because $\{\tilde{V}_j\}_{j \in \mathbb{Z}}$ has the same properties. Properties (c) and (d) are satisfied by (3.16). Since $\psi_\varepsilon \in \mathcal{R}_r(\mathbb{R}^n) \cap \mathcal{S}_r(\mathbb{R}^n)$, $\varepsilon \in E$, it follows that $\{V_j\}_{j \in \mathbb{Z}}$ is an r -regular multiresolution analysis of $L^2(\mathbb{R}^n)$ for multiwavelets. \square

The proof of Theorem 2 takes the structure of the multiresolution analysis into account.

4. CONSTRUCTION OF SPLIT-TYPE MULTIWAVELETS

In this section, we construct an r -regular multiresolution analysis of multiwavelets with $d = 2^n$ directly from an r -regular multiresolution analysis of uniwavelets and a $(2^n - 1)$ -fold regular multiresolution analysis of uniwavelets. This construction does not use the differential geometric Theorem 1 of [7].

Theorem 3. *Let an r -regular multiresolution analysis $\{\tilde{V}_j^0\}_{j \in \mathbb{Z}}$ and a $(2^n - 1)$ -fold regular multiresolution analysis $\{\tilde{V}_j^\ell\}_{j \in \mathbb{Z}}, \ell \in J_n \setminus \{0\}$, of uniwavelets in $L^2(\mathbb{R}^n)$ be given. Then there exists an r -regular family $\{\Psi_\varepsilon\}_{\varepsilon \in E}$ of $2^n - 1$ multiwavelets with $d = 2^n$.*

Proof. We need only construct $M(\xi) \in U(2^n d; C^\infty(\mathbb{T}^n))$. For a given r -regular multiresolution analysis $\{\tilde{V}_j^0\}_{j \in \mathbb{Z}}$ of uniwavelets in $L^2(\mathbb{R}^n)$, denote by $\tilde{\varphi}^0$ its scaling function and by $\tilde{\psi}_{\ell'}^0, \ell' \in J_n \setminus \{0\}$, its wavelets. For a given regular multiresolution analysis $\{\tilde{V}_j^\ell\}_{j \in \mathbb{Z}}, \ell \in J_n \setminus \{0\}$, of uniwavelets in $L^2(\mathbb{R}^n)$, denote by $\tilde{\varphi}^\ell$ its scaling function and by $\tilde{\psi}_{\ell'}^\ell, \ell' \in J_n \setminus \{0\}$, its wavelets, and define $\tilde{\psi}_0^\ell := \tilde{\varphi}^\ell, \ell \in J_n$. Then there exist $\tilde{m}_{\ell'}^\ell(\xi) \in C^\infty(\mathbb{T}^n), \ell \in J_n, \ell' \in J_n$, such that

$$\widehat{\tilde{\psi}_{\ell'}^\ell}(2\xi) = \tilde{m}_{\ell'}^\ell(\xi) \widehat{\tilde{\psi}_0^\ell}(\xi). \quad (4.1)$$

Define a multiscaling function by $\Phi = (\varphi_\delta)_{\delta \in D}^T := (\tilde{\psi}_{\ell'}^0)_{\ell' \in J_n}^T$ with $\delta = \ell' + 1$. Then we have

$$M_0(\xi) = (\tilde{m}_{\ell'}^0(\xi) \delta_{0, \ell''})_{(\ell', \ell'') \in J_n \times J_n}, \quad (4.2)$$

where $\delta_{\ell', \ell''}$ denotes the Kronecker delta. Define

$$M_\ell(\xi) := (\tilde{m}_{\ell'}^\ell(\xi) \delta_{\ell, \ell''})_{(\ell', \ell'') \in J_n \times J_n}, \quad \ell \in J_n \setminus \{0\}. \quad (4.3)$$

Then every element of $M_\ell(\xi)$ belongs to $C^\infty(\mathbb{T}^n)$. The orthonormality of the uniwavelet basis implies that

$$\sum_{\eta \in R} \tilde{m}_{\ell'}^\ell(\xi + \eta\pi) \overline{\tilde{m}_{\ell''}^\ell(\xi + \eta\pi)} = \delta_{\ell', \ell''}, \quad \ell, \ell', \ell'' \in J_n,$$

(see, for instance, [20, Lemma 7]). \square

Remark 5. Sometimes, for $r > 0$, it is advantageous to combine an r -regular multiresolution analysis of uniwavelets with a $(2^n - 1)$ -fold regular multiresolution analysis of uniwavelets (like the Haar wavelets) because, without losing the r -regularity and vanishing moments, we can have multiwavelets with shorter supports or shorter widths than those of the multiscaling functions, as can be seen in Figs. 2 and 3 and in Table 1.

Definition 7. An r -regular family $\{\Psi_\varepsilon\}_{\varepsilon \in E}$ of $2^n - 1$ multiwavelets with $d = 2^n$ constructed in Theorem 3 from an r -regular multiresolution analysis $\{\tilde{V}_j^0\}_{j \in \mathbb{Z}}$ and a $(2^n - 1)$ -fold regular multiresolution analysis $\{\tilde{V}_j^\ell\}_{j \in \mathbb{Z}}, \ell \in J_n \setminus \{0\}$, of uniwavelets is said to be of *split type*.

By using existing r -regular uniwavelets (for instance, those of Meyer, Daubechies, Battle–Lemarié, and so on) and Theorem 3, we can construct examples of r -regular multiwavelets of split type. In the next section, examples are given in the case $n = 1$.

5. ONE-DIMENSIONAL MULTIWAVELETS OF SPLIT TYPE

Let $n = 1$ and $d = 2^1 = 2$. Then, the construction in Theorem 3 has to be of split type.

In the following examples of one-dimensional r -regular multiwavelets of split type, we change notation from multiwavelets to uniwavelets.

We start by reviewing the construction of wavelets according to Daubechies [17]. Given a scaling function $\varphi(x)$, there exists a 2π -periodic function $m_0(\xi)$ such that the following two-scale relation

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi), \quad \text{where } m_0(\xi) \in L^2(\mathbb{T}),$$

holds. The wavelet $\psi(x)$ is obtained from the two-scale relation

$$\widehat{\psi}(2\xi) = m_1(\xi)\widehat{\varphi}(\xi), \quad \text{where } m_1(\xi) := -e^{-i\xi}\overline{m_0(\xi + \pi)}.$$

The low-pass and high-pass filters $m_0(\xi)$ and $m_1(\xi)$ are called the symbols, or masks, of $\varphi(x)$ and $\psi(x)$, respectively. If $\varphi(x)$ is compactly supported, then $m_0(\xi)$ is a trigonometric polynomial and $\psi(x)$ is compactly supported. For instance, in [17] Tables 6.1 and 6.3 list the coefficients $\{{}_N h_k\}$ of the symbols

$${}_N m_0(\xi) = 2^{-1/2} \sum_{k=0}^{2N-1} {}_N h_k e^{-ik\xi}, \quad (5.1)$$

with the length parameter $N \in \mathbb{N} \setminus \{0\}$ depending upon the regularity parameter r .

To construct r -regular multiwavelets of split type, we take an r -regular uniscaling function and uniwavelet pair, φ and ψ , and write

$$\Phi(x) = \begin{bmatrix} \varphi^1(x) \\ \varphi^2(x) \end{bmatrix} = \begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix}.$$

Next we set

$$\widehat{\Phi}(2\xi) = \begin{bmatrix} \widehat{\varphi}^1(2\xi) \\ \widehat{\varphi}^2(2\xi) \end{bmatrix} = M_0(\xi) \begin{bmatrix} \widehat{\varphi}(\xi) \\ \widehat{\psi}(\xi) \end{bmatrix}, \quad \text{where } M_0(\xi) = \begin{bmatrix} m_0(\xi) & 0 \\ m_1(\xi) & 0 \end{bmatrix}.$$

Finally, we need only find a matrix $M_1(\xi) \in \text{Mat}(2 \times 2; C^\infty(\mathbb{T}))$ such that

$$\begin{bmatrix} M_0(\xi) & M_0(\xi + \pi) \\ M_1(\xi) & M_1(\xi + \pi) \end{bmatrix} \in U(4; C^\infty(\mathbb{T})). \quad (5.2)$$

By the orthonormality of the uniwavelet basis, the symbols $m_0(\xi), m_1(\xi) \in C^\infty(\mathbb{T})$ satisfy the relations

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1, \quad (5.3)$$

$$|m_1(\xi)|^2 + |m_1(\xi + \pi)|^2 = 1, \quad (5.4)$$

$$m_0(\xi)\overline{m_1(\xi)} + m_0(\xi + \pi)\overline{m_1(\xi + \pi)} = 0. \quad (5.5)$$

Thus, (5.2) holds if and only if the matrix $M_1(\xi)$ is of the form

$$M_1(\xi) = \begin{bmatrix} 0 & \tilde{m}_0(\xi) \\ 0 & \tilde{m}_1(\xi) \end{bmatrix}, \quad (5.6)$$

where the functions $\tilde{m}_0(\xi), \tilde{m}_1(\xi) \in C^\infty(\mathbb{T})$ satisfy (5.3)–(5.5). Hence, if we take the symbols $\tilde{m}_0(\xi)$ and $\tilde{m}_1(\xi)$ of a regular scaling function $\tilde{\varphi}(x)$ and a regular wavelet $\tilde{\psi}(x)$, then (5.2) is obviously satisfied. These multiwavelets are said to be of *split type*. In this case, we have

$$\widehat{\Psi}(2\xi) = \begin{bmatrix} \widehat{\psi}^1(2\xi) \\ \widehat{\psi}^2(2\xi) \end{bmatrix} = M_1(\xi) \begin{bmatrix} \widehat{\varphi}(\xi) \\ \widehat{\psi}(\xi) \end{bmatrix} = \begin{bmatrix} \tilde{m}_0(\xi)\widehat{\psi}(\xi) \\ \tilde{m}_1(\xi)\widehat{\psi}(\xi) \end{bmatrix}. \quad (5.7)$$

Therefore $\psi^1(x) = 2^{1/2}\psi_{\text{split}}^1(2x)$ and $\psi^2(x) = 2^{1/2}\psi_{\text{split}}^2(2x)$, where ψ_{split}^1 and ψ_{split}^2 are split wavelets defined in [17, Section 10.5]. Thus, one-dimensional multiwavelets of split type are identical with split wavelets.

Given the above analysis filter pair $M_0(\xi)$ and $M_1(\xi)$, it is mentioned in Cooklev [10], p. 193, that perfect reconstruction is achieved by means of the synthesis filter pair

$$\begin{bmatrix} m_1(\xi + \pi) & -m_0(\xi + \pi) \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ \tilde{m}_1(\xi + \pi) & -\tilde{m}_0(\xi + \pi) \end{bmatrix}.$$

Remark 6. In the one-dimensional case, wavelet packets are defined in [17, Section 10.6] by the formula

$$\widehat{\psi}_{\ell; \varepsilon_1, \dots, \varepsilon_\ell}(2^\ell \xi) = \left[\prod_{j=1}^{\ell} m_{\varepsilon_j}(2^{\ell-j} \xi) \right] \widehat{\psi}(\xi), \quad (5.8)$$

where, at the j^{th} splitting, $\varepsilon_j = 0$ or 1 indicates the choice m_0 or m_1 . Hence wavelet packets with $\psi_{1;0}$ and $\psi_{1;1}$ coincide with our multiwavelets of split type ψ^1 and ψ^2 if $\tilde{m}_0 = m_0$ and $\tilde{m}_1 = m_1$.

If we take Daubechies' symbol ${}_N m_0(\xi)$ as $\tilde{m}_0(\xi)$, then the multiwavelets are easily constructed by the formulas

$$\psi^1(x) := 2^{1/2} \sum_{k=0}^{2N-1} {}_N h_k \psi(2x - k), \quad (5.9)$$

$$\psi^2(x) := 2^{1/2} \sum_{k=0}^{2N-1} (-1)^{k+1} {}_N h_k \psi(2x + k - 1). \quad (5.10)$$

This procedure has been applied with \tilde{m}_0 and \tilde{m}_1 of Haar and Daubechies D2 wavelets (shown in Fig. 1) and the results are reported below in Figs. 2 and 3 and Table 1.

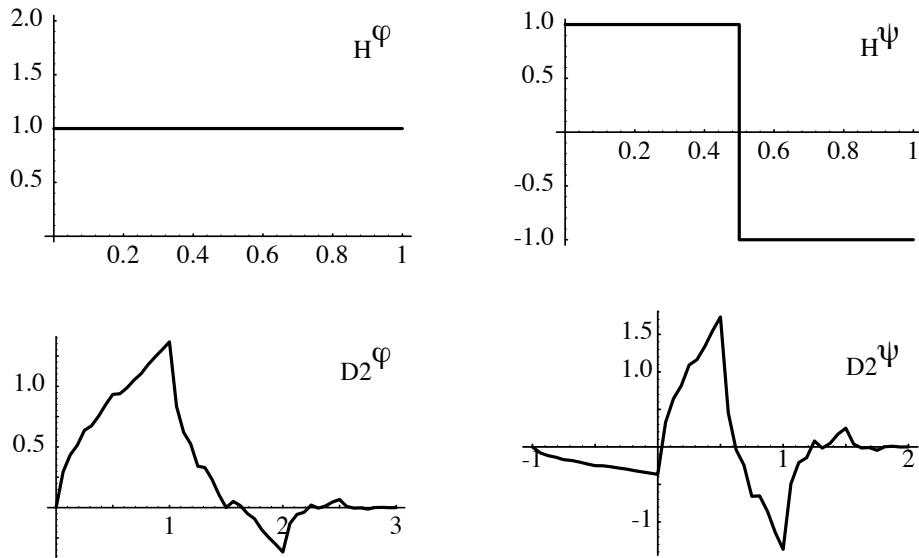


FIGURE 1. Haar and Daubechies D2 scaling and wavelets ${}_H\varphi$, ${}_H\psi$, and ${}_{D2}\varphi$, ${}_{D2}\psi$.

The first column in Figs. 2 and 3 shows the multiscaling functions φ^1 and φ^2 corresponding to Meyer's ∞ -regular scaling and wavelet ${}_M\varphi$ and ${}_M\psi$ [17, p. 15, Fig. 1.8], Daubechies' ${}_{D3}\varphi$ and ${}_{D3}\psi$ [17, p. 197, Fig. 6.3], the coiflets ${}_{C6}\varphi$ and ${}_{C6}\psi$ [17, p. 260, Fig. 8.3], and the symlets ${}_{S6}\varphi$ and ${}_{S6}\psi$ [17, p. 199, Fig. 6.4], respectively.

Coiflets and symlets are special wavelets which are defined as follows: *coiflets* of order L are orthonormal wavelet bases with vanishing moments for φ and ψ , from the first and zeroth to the $(L-1)^{\text{st}}$ moments, respectively,

$$\int \varphi(x) dx = 1, \quad \int \psi(x) dx = 0, \quad \int x^l \varphi(x) dx = 0, \quad \int x^l \psi(x) dx = 0, \\ l = 1, \dots, L-1,$$

and *symlets* are the “least asymmetric” compactly supported wavelets ψ with maximum number of vanishing moments.

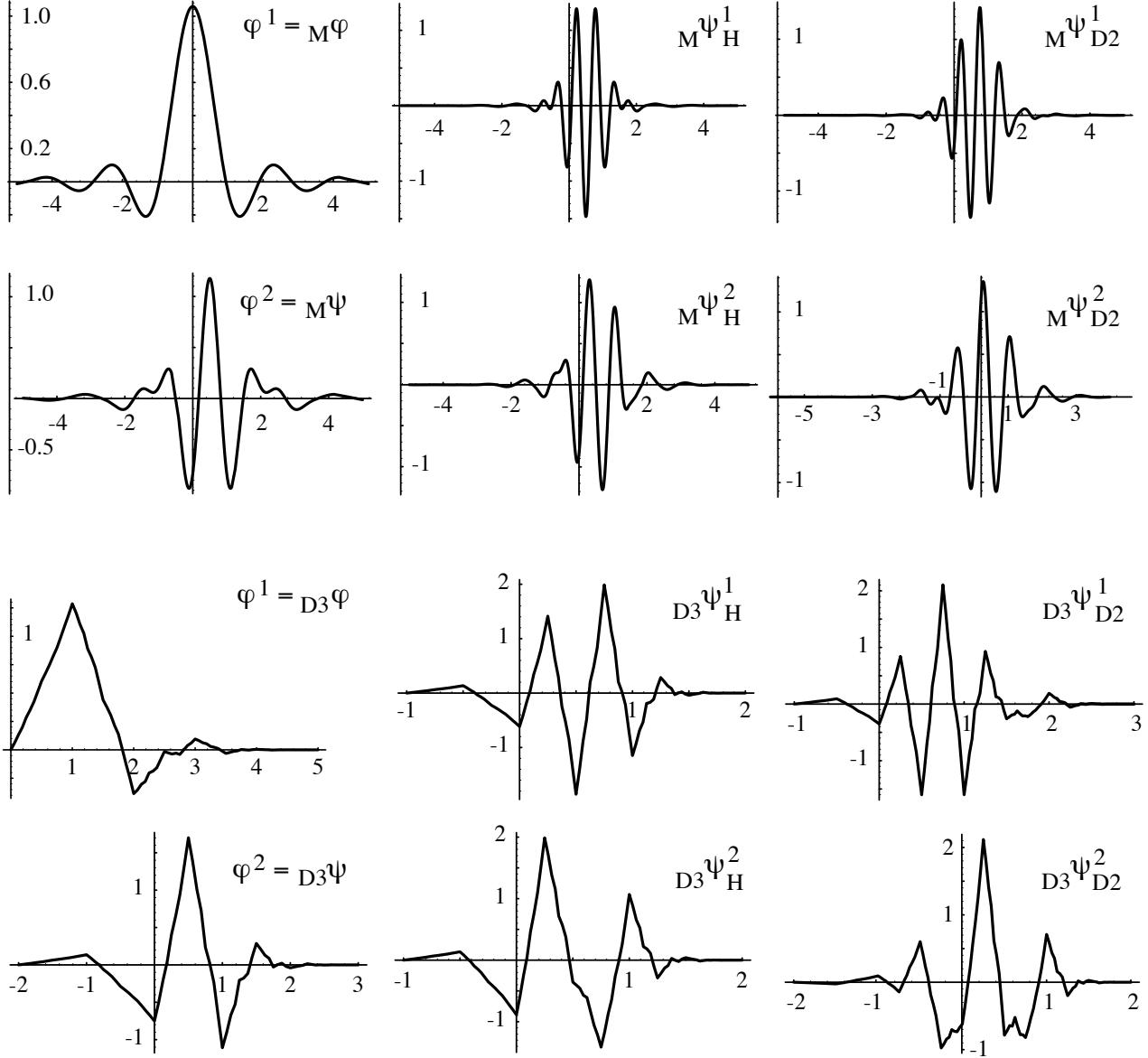


FIGURE 2. Multiscaling functions, φ^1 and φ^2 , (left) taken as Meyer's, $M\varphi$, $M\psi$ (top half), or Daubechies', $D_3\varphi$, $D_3\psi$ (bottom half), uniscaling functions and uniwavelets, respectively. The corresponding multiwavelets, ψ^1 and ψ^2 , are produced by means of Haar's (center) or Daubechies' D2 (right) low-pass and high-pass filters $m_0(\xi)$ and $m_1(\xi)$, respectively.

The second column of these figures shows the wavelets ψ_H^1 and ψ_H^2 constructed from φ^1 and φ^2 by means of the Haar symbols

$${}_H m_0(\xi) = \frac{1}{2} + \frac{1}{2} e^{-i\xi} \quad \text{and} \quad {}_H m_1(\xi) = \frac{1}{2} - \frac{1}{2} e^{-i\xi}.$$

Similarly, the third column shows the wavelets ψ_{D2}^1 and ψ_{D2}^2 constructed from φ^1

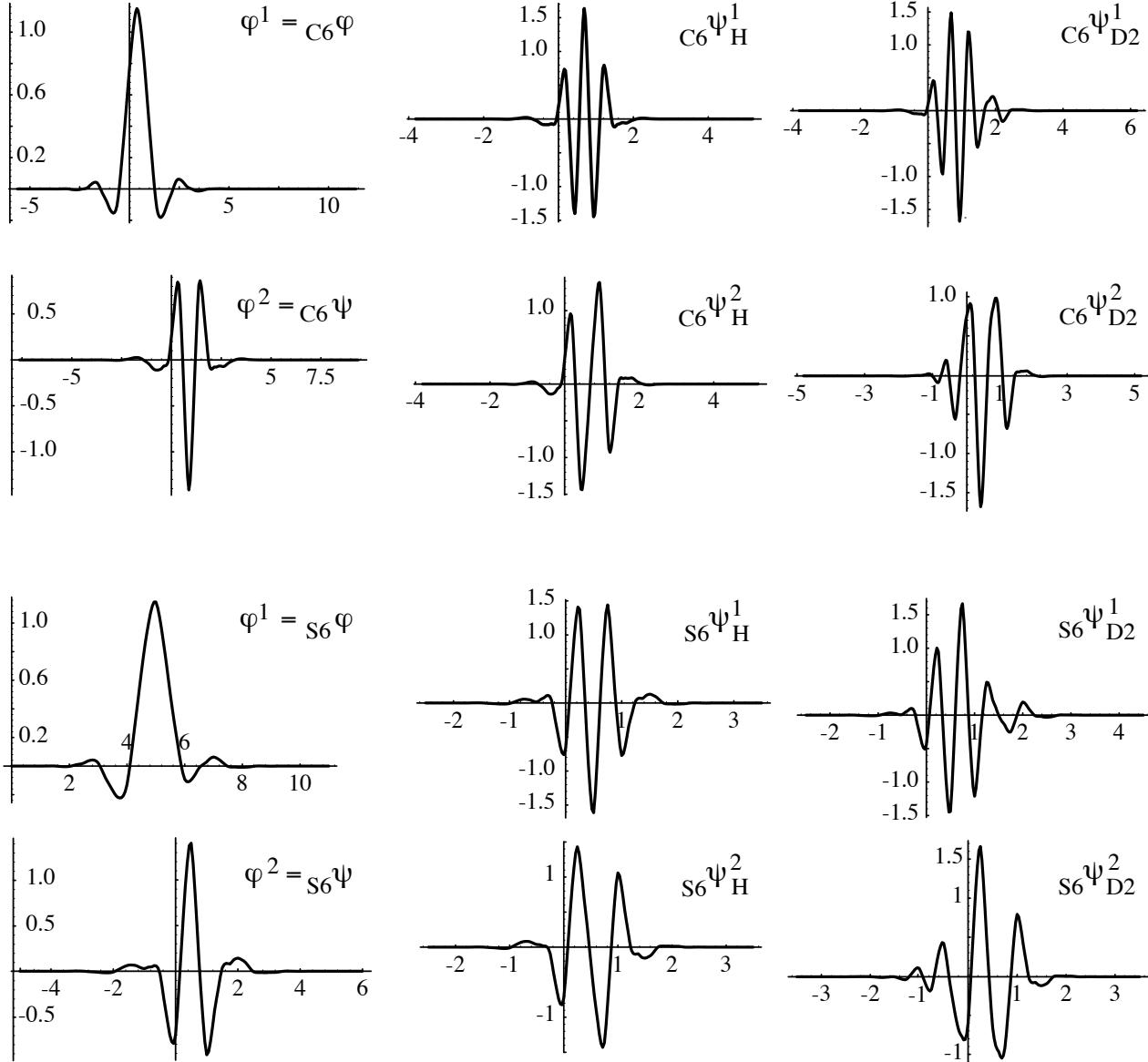


FIGURE 3. Multiscaling functions, φ^1 and φ^2 , (left) taken as C6 coiflets, $c_6\varphi$, $c_6\psi$ (top half), or S6 symlets, $s_6\varphi$, $s_6\psi$ (bottom half), uniscaling functions and uniwavelets, respectively. The corresponding multiwavelets, ψ^1 and ψ^2 , are produced by means of Haar's (center) or Daubechies' D2 (right) low-pass and high-pass filters $m_0(\xi)$ and $m_1(\xi)$, respectively.

and φ^2 by means of Daubechies' D2 symbols with coefficients

$$\frac{1 + \sqrt{3}}{8}, \quad \frac{3 + \sqrt{3}}{8}, \quad \frac{3 - \sqrt{3}}{8}, \quad \frac{1 - \sqrt{3}}{8}.$$

If we let

$$x_{\text{av}} := \frac{1}{\|f\|_2} \int_{-\infty}^{\infty} x |f(x)|^2 dx$$

denote the *average*, or center, of a function $f \in L^2(\mathbb{R})$, then the *width*, or diameter, of f is measured by the integral

$$\Delta_f := \frac{1}{\|f\|_2} \left[\int_{-\infty}^{\infty} (x - x_{\text{av}})^2 |f(x)|^2 dx \right]^{1/2}.$$

The width of split multiwavelets of Meyer type, and the length of the support of multiwavelets of Daubechies, coiflet and symlet types are listed in Table 1. In these cases, it is seen that the multiscaling functions φ^1 and φ^2 have longer support than the corresponding multiwavelets ψ^1 and ψ^2 .

TABLE 1. Widths and support lengths of multiscaling functions and multiwavelets of Meyer type and Daubechies D3, coiflet C6 and symlet S6 types, respectively, obtained by means of Haar's and Daubechies' D2 filters.

Meyer type Width	$M\varphi$ 0.583	$M\psi$ 0.714	$M\psi_H^1$ 0.381	$M\psi_H^2$ 0.485	$M\psi_{D2}^1$ 0.407	$M\psi_{D2}^2$ 0.504
Daubechies D3 type Support length	$D3\varphi$ 5	$D3\psi$ 5	$D3\psi_H^1$ 3	$D3\psi_H^2$ 3	$D3\psi_{D2}^1$ 4	$D3\psi_{D2}^2$ 4
Coiflet C6 type Support length	$C6\varphi$ 17	$C6\psi$ 17	$C6\psi_H^1$ 9	$C6\psi_H^2$ 9	$C6\psi_{D2}^1$ 10	$C6\psi_{D2}^2$ 10
Symlet S6 type Support length	$S6\varphi$ 11	$S6\psi$ 11	$S6\psi_H^1$ 6	$S6\psi_H^2$ 6	$S6\psi_{D2}^1$ 7	$S6\psi_{D2}^2$ 7

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