MICROLOCAL ANALYSIS AND MULTIWAVELETS

RYUICHI ASHINO
Mathematical Sciences, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan
E-mail: ashino@cc.osaka-kyoiku.ac.jp

CHRISTOPHER HEIL
School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332 USA
E-mail: heil@math.gatech.edu

MICHIHIRO NAGASE
Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail: nagase@math.wani.osaka-u.ac.jp

RÉMI VAILLANCOURT
Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5
E-mail: remi@uottawa.ca

Multiwavelets come with several scaling functions. Microlocal filtering is done with adapted orthonormal multiwavelets, which can be considered as the action of pseudodifferential operators whose symbols are characteristic functions of disjoint sets in Fourier space. Expansion of functions or signals in terms of an orthonormal multiwavelet basis gives a rough estimate of their microlocal content. Prefilters, which can be represented in terms of the n-D Hilbert transform, are designed and a fast algorithm is considered.
Keywords : multiwavelet, microlocal analysis, pseudodifferential operators, prefilter

1 Wavelets and Multiwavelets

An orthonormal multiwavelet basis is usually defined as an orthonormal wavelet basis generated by means of a multiresolution analysis from several scaling functions. By this definition, a family of wavelet functions can be divided into several groups associated to each scaling function. Each scaling function constructs a closed subspace and these subspaces give a decomposition of the hole space $L^2(\mathbb{R}^n)$ into an orthogonal sum. This means that multiwavelets can be though as a generalization of uniwavelets to vector valued functions. Here uniwavelets are generated by means of a multiresolution analysis from one scaling function. Our definition of multiwavlets extracts this structure of vectors as follows.

Definition 1 *Given $f \in L^2(\mathbb{R}^n)$, let $f_{jk}(x)$ denote the scaled and shifted function*

$$f_{jk}(x) = 2^{nj/2} f(2^j x - k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n.$$

Let D be a finite index set. A system $\{(\psi_\delta)_{jk}\}_{\delta\in D, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called an orthonormal wavelet basis and a system $\{\psi_\delta\}_{\delta\in D}$ is called a family of orthonormal wavelet functions if the system $\{(\psi_\delta)_{jk}\}_{\delta\in D, j\in\mathbb{Z}, k\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Moreover, if the cardinality of D is an integer multiple of $(2^n - 1)$, that is, $\operatorname{card} D = (2^n - 1)d$, $d \in \mathbb{N}$, *then the system $\{(\psi_\delta)_{jk}\}_{\delta\in D, j\in\mathbb{Z}, k\in\mathbb{Z}^n}$ is called an orthonormal multiwavelet basis and a vector of functions $\Psi = [\psi_\delta]_{\delta\in D}$ is called an* *orthonormal multiwavelet function.*

To show our main theorem, Theorem 3, we shall use the following Theorem 1 on a characterization of functions that generate wavelets and related expansions, which is essentially Theorem 1 in Frazier, Garrigós, Wang and Weiss[1].

Theorem 1 *Let $L \in \mathbb{N}$. Suppose $\{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, then*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\ell \in \{1,\ldots,L\},\, j\in\mathbb{Z},\, k\in\mathbb{Z}^n} \left|\left(f, \psi^\ell_{j,k}\right)\right|^2$$

for all $f \in L^2(\mathbb{R}^n)$ if and only if the functions $\{\psi^1, \psi^2, \ldots, \psi^L\}$ satisfy the following two equalities:

$$\sum_{\ell\in\{1,\ldots,L\},\, j\in\mathbb{Z}} |\widehat{\psi^\ell}(2^j\xi)|^2 = 1, \qquad a.a.\ \ \xi \in \mathbb{R}^n,$$

$$t_q(\xi) = 0, \qquad a.a.\ \ \xi \in \mathbb{R}^n,\ \ \forall q \in \mathbb{Z}^n \backslash (2\mathbb{Z})^n,$$

where

$$t_q(\xi) := \sum_{\ell\in\{1,\ldots,L\},\, j\in\mathbb{Z}_+} \widehat{\psi^\ell}(2^j\xi)\overline{\widehat{\psi^\ell}(2^j(\xi+2\pi q))}, \qquad \mathbb{Z}_+ := \mathbb{N}\cup\{0\}.$$

2 Microlocal Analysis

Our approach to microlocal analysis for Schwartz distributions is based on the theory of hyperfunctions, as introduced by Sato[2] and exposed in Kaneko[3] for the theory of linear partial differential equations with constant coefficients. A more complete treatment of microlocal filtering with multiwavelets can be found in Ashino, Heil, Nagase, and Vaillancourt[4].

Two important points are:

- Find directions along which a function can be continued analytically for every point $x \in \mathbb{R}^n$.
- A hyperfunction is defined as a sum of general boundary values of holomorphic functions in wedges whose edges are open subsets of $\mathbb{R}^n$.

2.1 Definition of n-D hyperfunctions

- A hyperfunction $f(x)$ is defined as a sum:

  $$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \qquad x \in \Omega,$$

  of boundary values

  $$F_j(x + i\Gamma_j 0) = \lim_{\substack{y\to 0\\ y\in\Gamma_j 0}} F_j(x+iy)$$

  of holomorphic functions $F_j(z)$ in infinitesimal wedges $\Gamma_j 0$ with edge $\Omega \subset \mathbb{R}^n$.

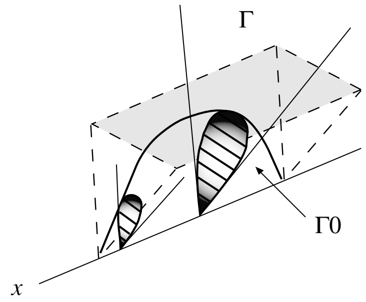

Figure 1. Infinitesimal wedge $\Gamma 0$.

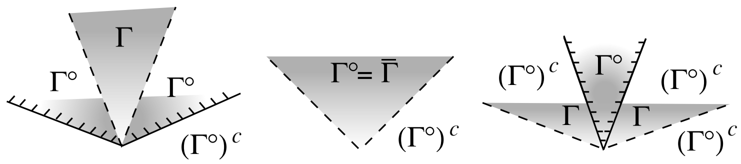

Figure 2. Open cone Γ, dual cone Γ°, and complement $(\Gamma^\circ)^c$ of dual cone.

2.2 Microanalyticity

To characterize the microanalyticity of a slowly increasing distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ by its Fourier transform $\hat{f}$, we introduce the *dual cone*, Γ°, of Γ, defined by

$$\Gamma^\circ := \{\xi \in \mathbb{R}^n; y \cdot \xi \geq 0 \text{ for every } y \in \Gamma\}.$$

Figure 2 shows examples of cones Γ, their dual cones Γ°, and their complements $(\Gamma^\circ)^c$.

Lemma 1 *Let Γ be an open convex cone. A slowly increasing distribution $f(x) \in \mathcal{S}'(\mathbb{R}^n)$ can be represented as the limit $f(x+i\Gamma 0)$ of a slowly increasing holomorphic function $f(z)$ in the infinitesimal wedge $\mathbb{R}^n + i\Gamma 0$ if and only if the Fourier transform $\hat{f}$ of f is exponentially decreasing in the open cone $(\Gamma^\circ)^c$, the complement of the dual cone Γ°, that is, $\hat{f}$ is exponentially decreasing on every closed proper subcone $\Gamma' \subset\subset (\Gamma^\circ)^c$.*

3 Microlocal Filtering

Our problems are the followings.

- How can we construct a suitable orthonormal multiwavelet function $\Psi = [\psi_\delta]_{\delta \in D}$ corresponding to each microanalytic direction $\mathbb{S}^{n-1}$?
- Is it possible to obtain information on the microlocal content of $f \in L^2(\mathbb{R}^n)$ from the wavelet coefficients $(f, (\psi_\delta)_{jk})$?
- Can orthonormal multiwavelet filtering separate microlocal contents?

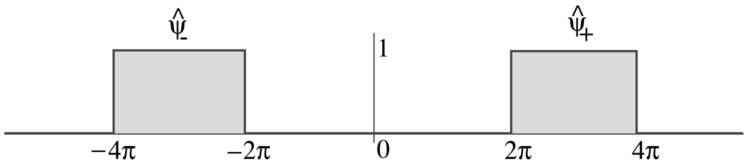

Figure 3. The Fourier transform of $\psi_\pm$.

We shall construct a suitable orthonormal multiwavelet basis which enables us to obtain information on the microlocal content of signals or functions. As this separation of microlocal contents can be explained by filtering, we call it *microlocal filtering*.

3.1 1-D orthonormal multiwavelets

Theorem 2 *Define $\psi_\pm$ by $\widehat{\psi}_\pm = \chi_{[\pm 2\pi, \pm 4\pi]}$ (Fig. 3). Then $\Psi := {}^t[\psi_+, \psi_-]$ is a multiwavelet function. Define the orthogonal projections $\mathcal{P}_\pm$ by*

$$\mathcal{P}_\pm f := \sum_{j,k \in \mathbb{Z}} (f, (\psi_\pm)_{jk})\ (\psi_\pm)_{jk}.$$

Then $\mathcal{P}_\pm f(x)$ can be extended analytically to $\{\operatorname{Im} z > 0\}$ and $\{\operatorname{Im} z < 0\}$, respectively.

- This orthonormal basis is found in Daubechies, Grossmann, and Meyer[5].

Define the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ by

$$H^2(\mathbb{R}_\pm) = \big\{ f \in L^2(\mathbb{R}) : \ \widehat{f}(\xi) = 0 \quad \text{a.a.} \ \ \xi \leq (\geq)\, 0 \big\}.$$

Then

$$L^2(\mathbb{R}) = H^2(\mathbb{R}_+) \oplus H^2(\mathbb{R}_-).$$

Each $\psi_\pm$ is a uniwavelet function of $H^2(\mathbb{R}_\pm)$, respectively.

- A tight frame similar to this orthonormal basis is known as a smooth frame for $H^2(\mathbb{R}_\pm)$ in Hernández and Weiss[6], section 8.4.
- In the n-dimensional case, the set of all microanalytic directions is $\mathbb{S}^{n-1}$, which is an infinite set.
- A generalization of Theorem 2 to the n-dimensional case will be given in Theorem 3.
- It is possible to tell fairly well in which directions f is microanalytic.
- The price to pay to get good angular resolution in $\mathbb{S}^{n-1}$ is the need for many multiwavelets.

3.2 n-D orthonormal multiwavelets

Notation 1 *We shall use the following notation in* $\mathbb{R}^n$.

- $\eta = (\eta_1, \ldots, \eta_n) \in H := \{\pm 1\}^n$, *parametrization of* 2^n *orthants in* $\mathbb{R}^n$.
- $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E := \{0,1\}^n \setminus \{0\}$, $2^n - 1$ *vertices of unit cube, less the origin.*
- $\varepsilon.*\eta := (\varepsilon_1\eta_1, \ldots, \varepsilon_n\eta_n)$.
- $Q_\eta := \prod_{k=1}^n [0, \eta_k]$, *unit cube, where* $[0,-1]$ *means* $[-1,0]$.
- $\mathcal{Q}_{j,\varepsilon,\eta} := \left\{\prod_{k=1}^n [\eta_k(\ell_k - 1), \eta_k \ell_k] + 2^j(\varepsilon.*\eta) : 1 \le \ell_1, \ldots, \ell_n \le 2^j,\ \ell_1, \ldots, \ell_n \in \mathbb{N},\ j \in \mathbb{Z}_+\right\}$.
- $\mathcal{Q} := \{Q_k\}_{k\in K}, \quad \iota(\mathcal{Q}) := \bigcup_{k \in K} Q_k$.
- $2\pi\mathcal{Q}_{j,\varepsilon,\eta} := \{2\pi Q\ ;\ Q \in \mathcal{Q}_{j,\varepsilon,\eta}\}$.
- $\mathbb{Z}_+^{E\times H}$ *is the set of all functions from* $E \times H$ *to* $\mathbb{Z}_+$.

Theorem 3 *Let* $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$. *For* $Q \in \mathcal{Q}_{j,\varepsilon,\eta}$, *define* ψ_Q *by*

$$\widehat{\psi}_Q = \chi_{2\pi Q},$$

where $\chi_{2\pi Q}$ *is the characteristic function of the cube* $2\pi Q$. *For* $\rho \in \mathbb{Z}_+^{E\times H}$, *let*

$$\mathcal{Q}_\rho := \bigcup_{(\varepsilon,\eta)\in E\times H} 2\pi\mathcal{Q}_{\rho(\varepsilon,\eta),\varepsilon,\eta}.$$

Then, $\Psi := [\psi_Q]_{Q\in\mathcal{Q}_\rho}$ *is an orthonormal wavelet function.*

In particular, if ρ *is a constant function, then* $\Psi := [\psi_Q]_{Q\in\mathcal{Q}_\rho}$ *is an orthonormal multiwavelet function.*

Proof. Let $\{\psi^1, \psi^2, \ldots, \psi^L\} = \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ and write $\Psi := [\psi_Q]_{Q\in\mathcal{Q}_\rho}$. Since it is easy to show the two equalities, we have

$$\|f\|^2 = \sum_{(Q,j,k)\in\mathcal{Q}_\rho\times\mathbb{Z}\times\mathbb{Z}^n} \left|\left(f, (\psi_Q)_{j,k}\right)\right|^2,$$

for all $f \in L^2(\mathbb{R}^n)$. Substitute f by $\psi_{\widetilde{Q}}$, $\widetilde{Q} \in \mathcal{Q}_\rho$, then

$$\|\psi_{\widetilde{Q}}\|^2 = \sum_{(Q,j,k)\in\mathcal{Q}_\rho\times\mathbb{Z}\times\mathbb{Z}^n} \left|\left(\psi_{\widetilde{Q}}, (\psi_Q)_{j,k}\right)\right|^2,$$

that is,

$$\|\psi_{\widetilde{Q}}\|^2(1 - \|\psi_{\widetilde{Q}}\|^2) = \sum_{(Q,j,k)\neq(\widetilde{Q},0,0)} \left|\left(\psi_{\widetilde{Q}}, (\psi_Q)_{j,k}\right)\right|^2 = 0,$$

which follows from the normality:

$$\|\,(\psi_Q)_{j,k}\,\|^2 = 1, \qquad (Q,j,k) \in \mathcal{Q}_\rho \times \mathbb{Z} \times \mathbb{Z}^n.$$

This completes the proof. □

Figure 4 shows how to get finer resolution in Fourier space.

Figure 5 illustrates 2-D multiwavelets given in Theorem 3.

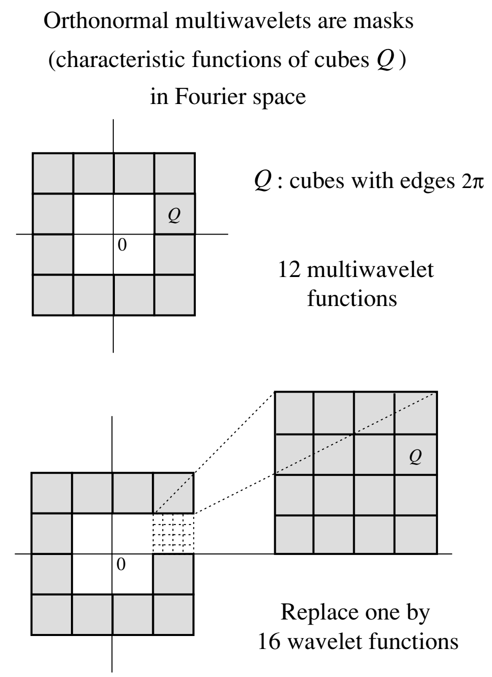

Figure 4. Technique for finer resolution in Fourier space.

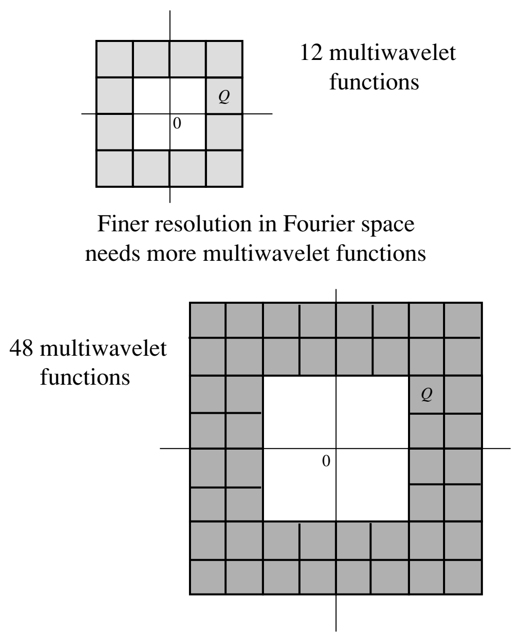

Figure 5. 2-D orthonormal multiwavelet functions in the Fourier space.

3.3 Pseudodifferential representation

Our microlocal filters can be represented by pseudodifferential operators. Let us explain the 2-D case corresponding to images.

Given an image $f(x, y)$ and a mask $p(\xi, \eta) = \chi_Q(\xi, \eta)$, filtering of f by p is represented by the pseudodifferential operator

$$\begin{aligned} Pf(x,y) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{i(x\xi+y\eta)} p(\xi,\eta)\hat{f}(\xi,\eta)\,d\xi\,d\eta \\ &= \frac{1}{4\pi^2}\int_{Q} e^{i(x\xi+y\eta)} \hat{f}(\xi,\eta)\,d\xi\,d\eta. \end{aligned}$$

Pseudodifferential operators, $P : f \mapsto Pf$ are pseudolocal operators, that is, they are not local operators but they do not spread or displace the singular support of f.

Micro-elliptic operators are studied in Hörmander[7]. Nonlinear heat operators, which are hypoelliptic operators, are used to denoise images in Mallat[8], Alvarez, Lions, and Morel[9].

4 Implementation of Microlocal Filtering

To implement the multiwavelet transform of f we need the scaling coefficients at high resolution. Recall that in the uniwavelet case, at very high resolution, the scaling functions are usually close to the delta function; hence the samples of the function f are used as scaling coefficients. However, for multiwavelets we need expansion coefficients for d scaling functions. Simply using nearby samples as scaling coefficients may be a bad choice. Data samples need to be preprocessed (*prefiltered*) to produce reasonable values for the expansion coefficients of scaling functions at the highest scale.

4.1 Prefilter design

Since our twelve multiwavelet functions are generated by four scaling functions (and other multiwavelet functions, such as the forty-eight multiwavelet functions in Figure 5), are generated by the basic twelve multiwavelet functions, these four scaling functions which generate the basic twelve multiwavelet functions can be used as prefilters for all our multiwavelets.

Our design of prefilter is the following. Define φ_η, $\eta \in H$ by $\widehat{\varphi}_\eta = \chi_{2\pi Q_\eta}$. Then, φ_η, $\eta \in H$ are the scaling functions for the twelve multiwavelet functions. Our prefilters are defined by

$$P_\eta = \mathcal{F}^{-1} \circ 2^{-nj_0}\chi_{2\pi 2^{j_0} Q_\eta} \circ \mathcal{F}, \qquad \eta \in H,$$

for sufficiently large j_0. Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively, and $2^{-nj_0}\chi_{2\pi 2^{j_0} Q_\eta}$ denotes the multiplication operator by the function $2^{-nj_0}\chi_{2\pi 2^{j_0} Q_\eta}$.

Our prefilters can be represented in terms of the n-D Hilbert transform, see Pandey and Singh[10].

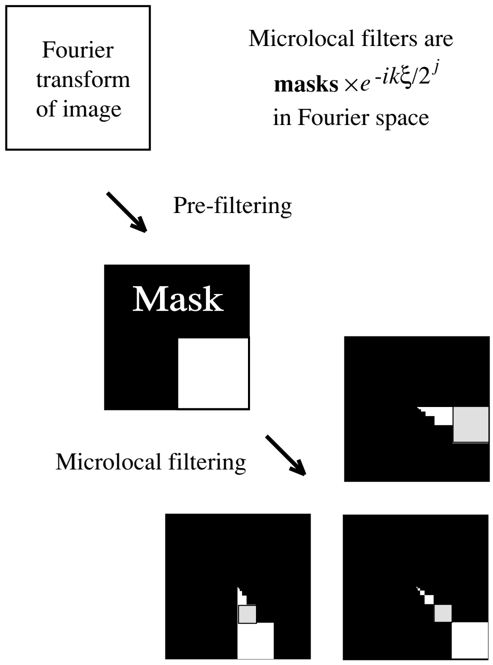

Figure 6. Multiwavelet masks of the fourth quadrant.

4.2 Two-dimensional masks

Figures 6 and 7 illustrate the prefiltering and filtering process of images in Fourier space.

4.3 Numerical example

Here is a numerical example of separation of singularities and reduction of noise. In a gray scale from 0 to 1, zero is white and 1 is black.

Figure 8 shows a “square flake” of height 1 rotated by 45 degrees and its discrete Fourier transform.

Figure 9 shows the “square flake” of height 1 with centered Gaussian of height 0.8 and the flake almost without the Gaussian obtained by adding the 4 diagonal parts of the Fourier transform less the 4 scaling functions. The Gaussian without the flake can be obtained by taking the 4 scaling functions and the 8 closest multiwavelets (the 4 corner wavelets are not included), that is, the corresponding pixels in the center of the Fourier transform.

Figure 10 shows the “square flake” with random noise of level 0.23 in the scale 0 to 1 and the partially denoised flake obtained by adding the 4 diagonal parts of the Fourier transform less the 4 scaling functions. Two-thirds of the noise have been removed.

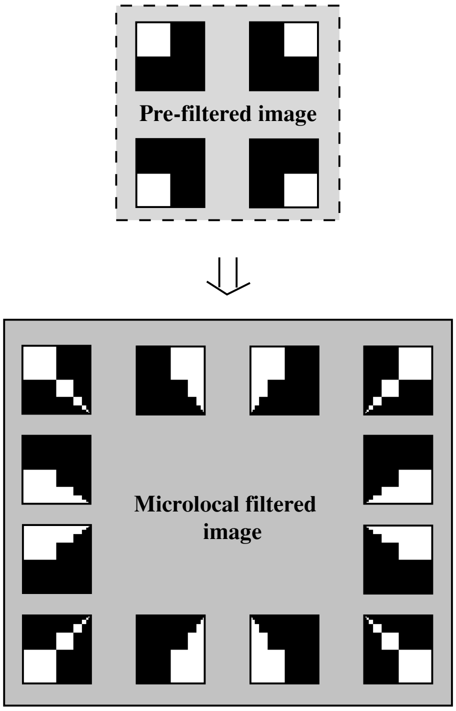

Figure 7. The twelve multiwavelet masks.

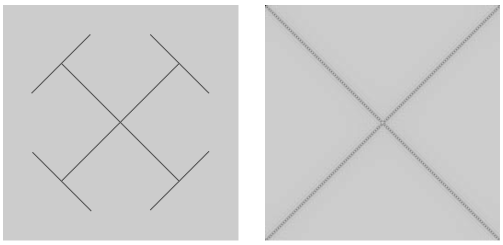

Figure 8. Original image of flake and Fourier transform.

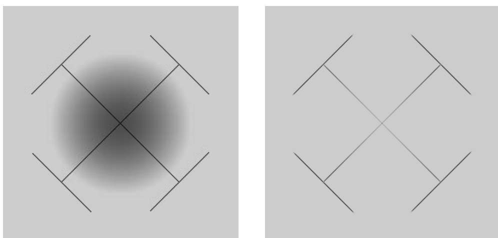

Figure 9. Flake with centered Gaussian (left) and sum of diagonal parts less scaling functions (right).

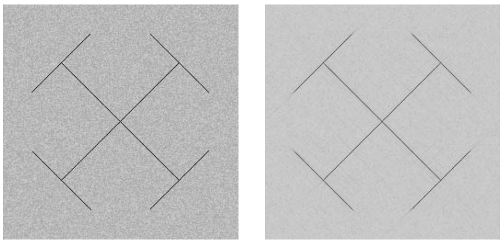

Figure 10. Noisy flake (left) and sum of diagonal parts less scaling functions (right).

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