SOME TOPICS ON WAVELETS

RYUICHI ASHINO

1. Introduction

Let us consider music. If the music is recorded from a live broadcast onto tape, we have a signal, that is, a function $f(x)$. The time-frequency analysis of music can be said to be making a music note.

Figure 1. Time-frequency analysis of music — note

If we want to change the music, first we edit the note, say, change from "do" to "re". Next we need to ask a music player to play the music according to the edited note. This procedure can be translated as follows. First decompose the signal into a sum of "time-frequency atoms". Next change some coefficients of time-frequency atoms and reconstruct a new signal from new coefficients. Making a music note corresponds to decomposition of a signal into time-frequency atoms. A music player corresponds to a reconstruction formula from time-frequency atoms.

The Fourier transform decomposes a signal into a sum of sine or cosine waves, the windowed Fourier transform decomposes a signal into a sum of translates of localized sine or cosine waves by multiplying a window function, and the wavelet transform decomposes a signal into a sum of scaled and translated waves with a constant shape. Those are illustrated in Figure 2.

In windowed Fourier analysis, the size of the window is fixed and the number of oscillations varies. A small window is "blind" to low frequencies, which are too large for the window. But if one uses a large window, information about a quick change will be lost in the information concerning the entire interval corresponding to the window. A "wavelet" is stretched or compressed to change the size of the window. This makes it possible to analyze a signal at different scales. The wavelet transform is sometimes called a "mathematical microscope". Big wavelets give an approximate image of the signal, while smaller wavelets zoom in on small details.

2. Preliminaries

Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}[f] = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx$$

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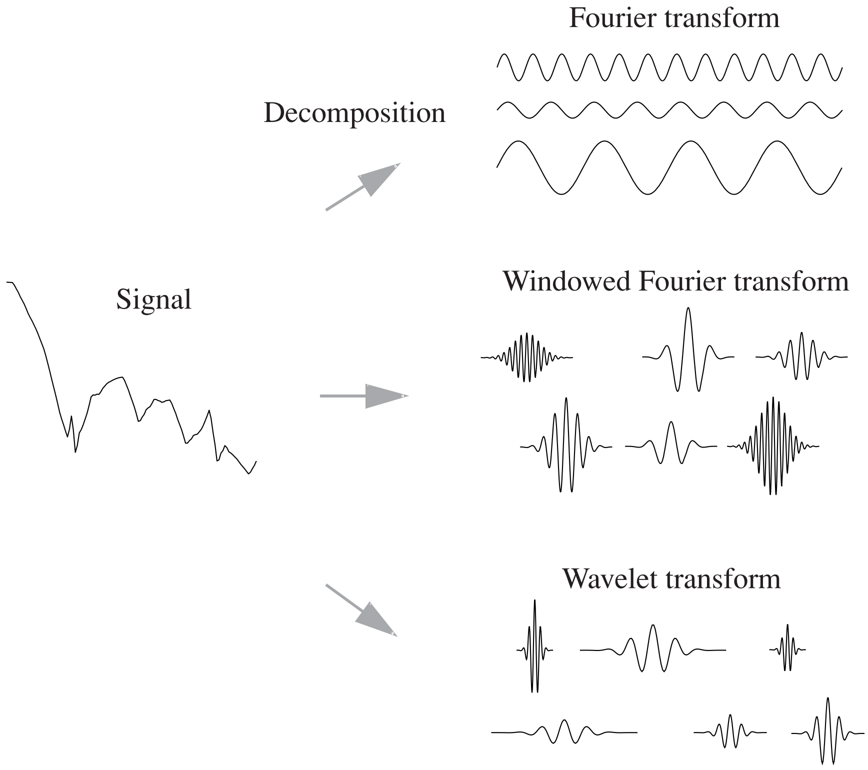

FIGURE 2. Fourier, windowed Fourier, and wavelet transforms.

and the inverse Fourier transform of a function $g \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}^{-1}[g] = \check{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\, g(\xi)\, d\xi.$$

There are two main principles in time-frequency analysis as follows.

- **Duality:**

$$\begin{aligned} \text{Smoothness of } f \quad &\Longleftrightarrow \quad \text{Decay at infinity of } \widehat{f}. \\ \text{Differentiation of } f \quad &\Longleftrightarrow \quad \text{Multiplication by polynomials to } \widehat{f}. \\ \text{Translation of } f \quad &\Longleftrightarrow \quad \text{Modulation of } \widehat{f}. \end{aligned}$$

- **Uncertainty:**

$$\text{It is impossible to localize both } f \text{ and } \widehat{f}.$$

One of keys of time-frequency analysis in the Hilbert space $L^2(\mathbb{R}^n)$ is Parseval's formula:

$$\langle f, g\rangle = (2\pi)^{-n}\langle \widehat{f}, \widehat{g}\rangle.$$

Notation 1.

$$\begin{aligned} &T_y f(x) = f(x-y), && \text{Translation operator,} \\ &M_\xi f(x) = e^{ix\xi}\, f(x), && \text{Modulation operator,} \\ &D_\rho f(x) = |\rho|^{-n/2} f(\rho^{-1}x), \quad \rho \in \mathbb{R}\backslash\{0\}, && \text{Dilation operator.} \end{aligned}$$

Lemma 1. *Each of these three operators T_y, M_ξ, D_ρ is unitary on $L^2(\mathbb{R}^n)$. That is, it is a surjection preserving the inner product of $L^2(\mathbb{R}^n)$:*

$$\|T_y f\| = \|f\|, \qquad \|M_\xi f\| = \|f\|, \qquad \|D_\rho f\| = \|f\|.$$

Then, the adjoint operators of these three operators are given by their inverses, respectively:

$$\langle T_y f, g\rangle = \langle f, T_{-y} g\rangle, \qquad \langle M_\xi f, g\rangle = \langle f, M_{-\xi} g\rangle, \qquad \langle D_\rho f, g\rangle = \langle f, D_{1/\rho} g\rangle.$$

Lemma 2.

$$\begin{aligned} T_y M_\xi &= e^{-i\xi y} M_\xi T_y, & M_\xi T_y &= e^{i\xi y} T_y M_\xi, \\ T_y D_\rho &= D_\rho T_{y/\rho}, & D_\rho T_y &= T_{\rho y} D_\rho, \\ M_\xi D_\rho &= D_\rho M_{\rho\xi}, & D_\rho M_\xi &= M_{\xi/\rho} D_\rho. \end{aligned}$$

Lemma 3.

$$\mathcal{F}[T_y f] = M_{-y}\mathcal{F}[f], \qquad \mathcal{F}[M_\omega f] = T_\omega \mathcal{F}[f], \qquad \mathcal{F}[D_\rho f] = D_{1/\rho}\mathcal{F}[f].$$

Denote $\mathbb{R}_+ := \{x \in \mathbb{R} \ ; \ x > 0\}$. Since both dilation D_a, $a \in \mathbb{R}_+$ and translation T_b, $b \in \mathbb{R}^n$ are unitary, the composition operator $T_b D_a$ is also unitary and called *time-scale operator.*

Lemma 4.

$$(T_b D_a)(T_x D_s) = T_{ax+b} D_{as}, \qquad b, x \in \mathbb{R}^n, \quad a, s \in \mathbb{R}_+.$$

3. Windowed Fourier Transform

The uncertainty principle states that the energy spread of a function and its Fourier transform cannot be simultaneously arbitrarily small. Using translations and modulations, D. Gabor[10] defined time-frequency atoms $\{M_\xi T_b g\}$ from a function g called *window function.* Since translations and modulations are unitary, the composition operator $M_\xi T_b$ is also unitary and called a *time-frequency shift.*

Definition 1. Fix a function $g \in L^2(\mathbb{R}^n)$, which is be called a *window function.* The *windowed Fourier transform* of $f \in L^2(\mathbb{R}^n)$ is defined by

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\overline{g(x-b)}\, dx &= \langle f(x), e^{ix\xi} g(x-b)\rangle \\ &= \langle f(x), M_\xi T_b g(x)\rangle. \end{aligned}$$

Let us consider one dimensional case $n = 1$. For a window function $g \in L^2(\mathbb{R})$, denote

$$x^* := \frac{1}{||g||^2}\int_{\mathbb{R}} x|g(x)|^2\, dx,$$

which is called the *center.* The energy of $M_\omega T_b g$ is concentrated in the neighborhood of $x^* + b$ over an interval of size Δ_g, measured by the standard deviation of $|(g/||g||)|^2$, that is,

$$\Delta_g := \frac{1}{||g||}\left(\int_{\mathbb{R}} (x - x^*)^2 |g(x)|^2\, dx\right)^{1/2},$$

which is called the *width.* Since the Fourier transform of $M_\omega T_b g$ is

$$\mathcal{F}[M_\omega T_b g](\xi) = T_\omega M_{-b}\hat{g} = \hat{g}(\xi - \omega)\, e^{-ib(\xi-\omega)}, \tag{1}$$

the energy of $\mathcal{F}[M_\omega T_b g]$ is therefore localized near the frequency $\xi^* + \omega$, over an interval of size $\Delta_{\hat{g}}$, which measures the domain where $\hat{g}(\xi)$ is not negligible. In a time-frequency plane (x, ξ), the energy spread of the atom $M_\omega T_b g$ is symbolically represented by the *time-frequency window*, which is defined as a rectangle centered at $(x^* + b, \xi^* + \omega)$ having a time width Δ_g and a frequency width $\Delta_{\hat{g}}$. We illustrate two time-frequency windows in Figure 3.

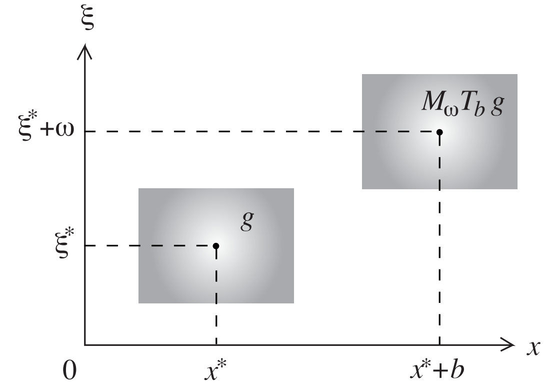

FIGURE 3. Time-frequency windows of two Gabor atoms g and $M_\omega T_b g$.

The uncertainty principle proves that

$$\Delta_g\, \Delta_{\hat{g}} \geq \frac{1}{2}.$$

The equality is satisfied if g is a Gaussian, in which case the atoms $M_\omega T_b g$ are called *Gabor functions*.

In applications, we want to look at signals in short time when they change quickly and in long time when they change slowly. But the shapes of time-frequency windows of $M_\omega T_b g$ are the same, the atoms $M_\omega T_b g$ are not enough. One way to overcome is introduce a scale parameter, that is, to use dilations. If we replace the window function g by $D_a g$, $a \in \mathbb{R}\backslash\{0\}$, then the centers of $D_a g$ and its Fourier transform are ax^* and ξ^*/a, respectively and the widths of $D_a g$ and its Fourier transform are $|a|\Delta_g$ and $\Delta_{\hat{g}}/|a|$, respectively. We illustrate two time-frequency windows in Figure 4.

Thus the windows of $M_\omega T_b(D_a g)$ change their shapes preserving their areas and move around in the time-frequency plane according to parameters a and (b, ω).

4. Uncertainty Principle

There are various kinds of formulations of uncertainty principles. For example, see V. Havin and B. Jöricke[14]. Here we give a formulation by D. Donoho and P. Strak[5].

Definition 2. Let T be a measurable set of $\mathbb{R}^n$ and denote by T^c, the complement of T in $\mathbb{R}^n$. For a positive ε, a function $f \in L^2(\mathbb{R}^n)$ is said to be ε-*concentrated* on T if

$$\left(\int_{T^c} |f(x)|^2\, dx \right)^{1/2} \leq \varepsilon \|f\|$$

is satisfied.

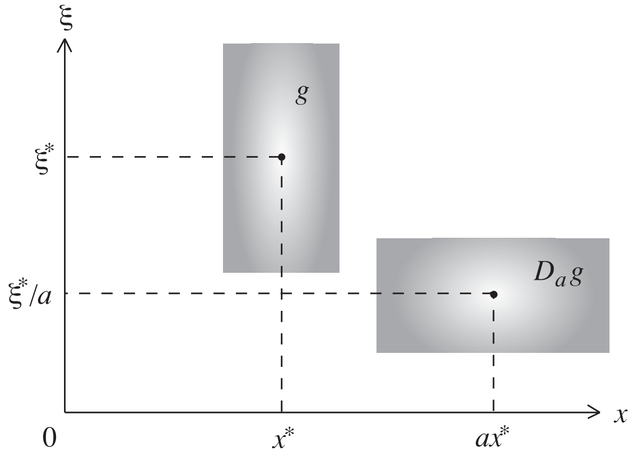

FIGURE 4. Time-frequency windows of two window function g and $D_a g$.

Theorem 1. *Let T, Ω be measurable sets of $\mathbb{R}^n$. For two positive numbers ε_T, ε_Ω satisfying $\varepsilon_T + \varepsilon_\Omega \le 1$, assume that $f \in L^2(\mathbb{R}^n)\backslash\{0\}$ is ε_T-concentrated on T and $\widehat{f}$ is ε_Ω-concentrated on Ω. Then,*

$$|T| \cdot |\Omega| \ge (2\pi)^n (1 - \varepsilon_T - \varepsilon_\Omega)^2,$$

where $|T|$, $|\Omega|$ are the measures of T, Ω.

5. Wavelet Transform

To analyze signal structures of various scales, it is necessary to use time-frequency atoms with different time supports. As we mentioned before, we want to look at signals in short time when they change quickly and in long time when they change slowly. The windows of $M_\omega T_b D_a g$ would be enough for time-frequency localizations, but they are too many. One way to reduce this kind of redundancy is the wavelet transform. In the wavelet transform, short time or long time are automatically chosen in time-frequency localizations according to changing speed of phenomena.

Lemma 2 implies that

$$M_\omega T_b D_a g = e^{i\omega b}\, T_b M_\omega D_a g = e^{i\omega b}\, T_b D_a M_{a\omega} g = T_b D_a (e^{i\omega b}\, M_{a\omega} g).$$

Here, $e^{i\omega b}\, M_{a\omega} g$ are localized waves. If we replace these localized waves by only one wave ψ, which will be called a wavelet function or wavelet, a family of dilates and translates $T_b D_a \psi$ of the function ψ may reduce the redundancy which the system of functions $M_\omega T_b D_a g$ has. These dilates and translates will be the atoms for time-frequency analysis. The wavelet transform decomposes signals over dilated and translated wavelets.

Definition 3. Fixed a function $\psi \in L^2(\mathbb{R}^n)$, which will be called a *wavelet function.* The *continuous wavelet transform* of $f \in L^2(\mathbb{R}^n)$ by a wavelet function ψ is defined by

$$W_\psi f(b, a) := |a|^{-n/2} \int_{\mathbb{R}^n} f(x) \overline{\psi\Big(\frac{x-b}{a}\Big)}\, dx = \langle f, T_b D_a \psi \rangle.$$

The function $W_\psi f(b, a)$ or its value at a point (b, a) is called a *wavelet coefficient.*

Remark 1. The continuous wavelet transform can be represented as

$$\langle f, T_b D_a \psi\rangle = \langle f, D_a T_{b/a} \psi\rangle.$$

Theorem 2. *Let $\psi \in L^2(\mathbb{R}^n)$ satisfy*

$$\int_{\mathbb{R}_+} |\widehat{\psi}(a\omega)|^2 \frac{da}{|a|} < +\infty, \qquad a.e.\ \omega \in \mathbb{S}^{n-1} \tag{2}$$

and there exists a positive constant K independent of $\omega \in \mathbb{S}^{n-1}$ such that

$$\int_{\mathbb{R}_+} |\widehat{\psi}(a\omega)|^2 \frac{da}{|a|} = K, \qquad a.e.\ \omega \in \mathbb{S}^{n-1} := \{\omega \in \mathbb{R}^n\ ;\ |\omega| = 1\}. \tag{3}$$

Then, for each $f \in L^2(\mathbb{R}^n)$,

$$f = K^{-1} \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^n} W_\psi f(b,a)\, T_b D_a \psi\, db \right) \frac{da}{|a|^{n+1}}. \tag{4}$$

Definition 4. The condition (2) and (3) on a function ψ is called an *admissibility condition.* The equality (4) is called the *inverse continuous wavelet transform.*

Let us consider one dimensional case $n = 1$. A function $\psi \in L^2(\mathbb{R})$ with a zero average:

$$\int_{\mathbb{R}} \psi(x)\, dx = 0 \tag{5}$$

satisfies the condition (2) if $\widehat{\psi}(\xi)$ is continuous near $\xi = 0$, because

$$\widehat{\psi}(0) = \int_{\mathbb{R}} \psi(x)\, dx = 0,$$

which removes the singurality at the origin in the integral of (2).

In time, $T_b D_a \psi$ is centered at $ax^* + b$ with a spread proportional to $|a|$. Its Fourier transform is

$$\mathcal{F}[T_b D_a \psi](\xi) = M_{-b} D_{1/a} \hat{\psi}(\xi) = e^{-ib\xi} \sqrt{|s|}\, \hat{\psi}(a\xi),$$

where $\hat{\psi}$ is the Fourier transform of ψ. In the time-frequency plane, a wavelet atom $T_b D_a \psi$ is symbolically represented by a rectangle centered at $(ax^* + b, \xi^*/a)$. The time and frequency spread are respectively proportional to $|a|$ and $1/|a|$. When a varies, the height and width of the rectangle change but its area remains constant, as illustrated by Figure 5.

6. Discrete Wavelet Transform

For numerical computations, let discretize a and b as $a_j,\ j \in \mathbb{Z}$ and $b_k,\ k \in \mathbb{Z}^n$ in (4). Since the integration can be approximated by its Riemann sum, we may expect

$$f(x) = \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} \langle f, T_{b_k} D_{a_j} \psi\rangle\, T_{b_k} D_{a_j} \widetilde{\psi}(x), \tag{6}$$

with a properly chosen $\widetilde{\psi}$. Put $c = b/a$ and reorder $\{b_k/a_j\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ as $\{c_\ell\}_{\ell\in\mathbb{Z}^n}$. Since $T_b D_a = D_a T_c$, (6) can be represented as

$$f(x) = \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} \langle f, D_{a_j} T_{c_\ell} \psi\rangle\, D_{a_j} T_{c_\ell} \widetilde{\psi}(x). \tag{7}$$

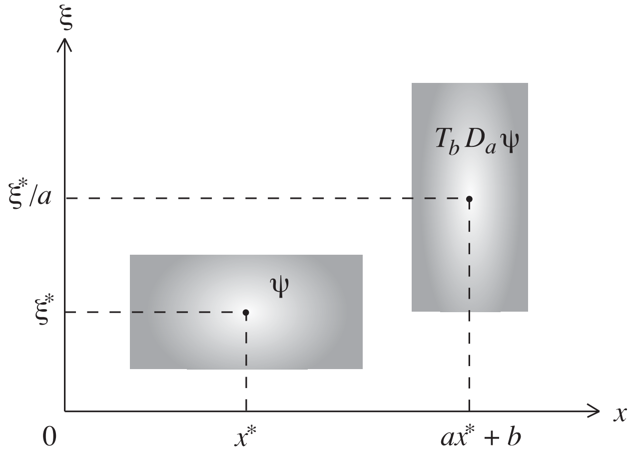

FIGURE 5. Time-frequency windows of two wavelets ψ and $T_b D_a \psi$.

If we put

$$\psi_{j,\ell} := D_{a_j} T_{c_\ell} \psi, \qquad \widetilde{\psi}_{j,\ell} := D_{a_j} T_{c_\ell} \widetilde{\psi},$$

then we have

$$f(x) = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \psi_{j,\ell}\rangle\, \widetilde{\psi}_{j,\ell}, \tag{8}$$

which is a kind of reconstruction formula of f using two families $\left\{\psi_{j,\ell}\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\left\{\widetilde{\psi}_{j,\ell}\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$. This is a special case of frames which will be defined below.

Frame theory was originally developed by R. Duffin and A. Schaeffer[6] to reconstruct band-limited signals f from irregularly spaced samples $\{f(t_n)\}_{n\in\mathbb{Z}}$. A function f is said to be band limited if its Fourier transform is supported in a finite interval $[-\pi/T, \pi/T]$, they were motivated by the classical Shannon sampling theorem, which asserts that

$$f(t_n) = \frac{1}{T}\langle f(t), h_T(t-t_n)\rangle, \qquad h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

They discussed general conditions under which one can recover a vector f in a separable Hilbert space $\mathcal{H}$ from its inner products $\langle f, \phi_n\rangle$ with a family of vectors $\{\phi_n\}_{n\in\mathbb{J}}$, where the index set $\mathbb{J}$ is a countable set, for example, $\mathbb{N}$, $\mathbb{Z}$, or a finite set.

Definition 5. A sequence $\{\phi_n\}_{n\in\mathbb{J}}$ is called a *frame* of $\mathcal{H}$ if there exist two constants $A > 0$ and $B > 0$ such that for any $f \in \mathcal{H}$

$$A\,\|f\|^2 \le \sum_{n\in\mathbb{J}} |\langle f, \phi_n\rangle|^2 \le B\,\|f\|^2.$$

The constants A and B are called *frame bounds*. A frame is said to be *tight* if $A = B$. A tight frame is said to be *Parseval* if $A = B = 1$. The operator $L : \mathcal{H} \mapsto \mathcal{H}$ defined by

$$Lf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \phi_n, \qquad \forall f \in \mathcal{H},$$

is called the *frame operator*.

Denote

$$\ell^2(\mathbb{J}) := \{x \ : \ \|x\|^2_{\ell^2(\mathbb{J})} := \sum_{n\in\mathbb{J}} |x[n]|^2 < +\infty\}.$$

The definition of frame gives an energy equivalence to invert the operator $U : \mathcal{H} \mapsto \ell^2(\mathbb{J})$ defined by

$$Uf[n] = \langle f, \phi_n\rangle, \qquad \forall n \in \mathbb{J}.$$

Denote by U^* the *adjoint* of U: $\langle Uf, x\rangle = \langle f, U^* x\rangle$. Then the frame operator L can be represented as

$$Lf = U^*Uf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \phi_n.$$

The system $\{\widetilde{\phi}_n\}_{n\in\mathbb{J}}$ defined by

$$\widetilde{\phi}_n = L^{-1}\phi_n \ = (U^*U)^{-1}\phi_n$$

is called the *dual frame* of $\{\phi_n\}_{n\in\mathbb{J}}$. If the frame is tight (i.e., $A = B$), then $\widetilde{\phi}_n = A^{-1}\,\phi_n$. The dual frame satisfies the inequalities

$$\frac{1}{B}\,\|f\|^2 \le \sum_{n\in\mathbb{J}} |\langle f, \widetilde{\phi}_n\rangle|^2 \le \frac{1}{A}\,\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

Let $\operatorname{ran} U$ denote the range of U, that is, the space of all Uf with $f \in \mathcal{H}$. If $\{\phi_n\}_{n\in\mathbb{J}}$ is a frame whose vectors are linearly dependent, then $\operatorname{ran} U$ is strictly included in $\ell^2(\mathbb{J})$ and U admits an infinite number of left inverses $\bar{U}^{-1}$:

$$\bar{U}^{-1}Uf = f, \qquad \forall f \in \mathcal{H}.$$

The left inverse that is zero on $\operatorname{ran} U^\perp$ is called the *pseudo-inverse* of U and is denoted by $\widetilde{U}^{-1}$:

$$\widetilde{U}^{-1}x = 0, \qquad \forall x \in \operatorname{ran} U^\perp.$$

In infinite dimensional spaces, the pseudo-inverse $\widetilde{U}^{-1}$ of an injective operator is not necessarily bounded. This induces numerical instabilities when trying to reconstruct f from Uf. The pseudo-inverse can be expressed in the form

$$\widetilde{U}^{-1} = (U^*U)^{-1}U^*$$

and

$$f = \widetilde{U}^{-1}Uf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \widetilde{\phi}_n = \sum_{n\in\mathbb{J}} \langle f, \widetilde{\phi}_n\rangle\, \phi_n.$$

When the frame is tight (i.e., $A = B$), as $\widetilde{\phi}_n = A^{-1}\,\phi_n$,

$$f = \widetilde{U}^{-1}Uf = \frac{1}{A}\sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \phi_n.$$

In this case, replacing ϕ_n by $\phi_n/\sqrt{A}$, without loss of generality we may assume that the frame bound is one, that is, $A = 1$.

Since the dual of a tight frame is a constant multiple of the frame itself, recovering functions from their frame coefficients does not require the computation of the dual frame. Hereafter, we shall focus on tight wavelet frames.

Given $f \in L^2(\mathbb{R}^n)$, let $f_{jk}(x)$ denote the scaled and shifted function

$$f_{jk}(x) := D_{1/2^j}T_k f = 2^{nj/2} f(2^j x - k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n. \tag{9}$$

In the case of general n dimension, the frequency ξ is n variables, on the contrary, the scale j is one variable. By this disagree, the time-frequency localization of $\{\psi_{j,\ell}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ of (8) is not enough accurate in frequency because the Fourier transform of wavelet function $\widehat{\psi}$ can localize only certain direction. To overcome this difficulty, we will use several wavelet functions which have different localizations in frequency.

Let $\mathbb{L}$ be a finite index set. A system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L},j\in\mathbb{Z},k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called a *tight wavelet frame* with frame bound A if

$$f(x) = \frac{1}{A}\sum_{\ell\in\mathbb{L},j\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \psi^\ell_{jk}\rangle\, \psi^\ell_{jk}(x), \qquad \forall f \in L^2(\mathbb{R}^n). \tag{10}$$

We recall that a system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L},j\in\mathbb{Z},k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called an *orthonormal wavelet basis* if it is an orthonormal basis for $L^2(\mathbb{R}^n)$. This is equivalent to saying that the system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L},j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is a tight wavelet frame with frame bound 1 and $\|\psi^\ell\|_{L^2(\mathbb{R}^n)} = 1$ for $\ell \in \mathbb{L}$.

The following general theorem which is essentially Theorem 1 stated and proved in M. Frazier, G. Garrigós, K. Wang and G. Weiss[7] for $\mathbb{R}^n$, gives necessary and sufficient conditions to have a tight wavelet frame in $\mathbb{R}^n$ with frame bound 1.

Theorem 3. *Suppose $\psi^\ell \subset L^2(\mathbb{R}^n)$ for $\ell \in \mathbb{L}$, then*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\ell\in\mathbb{L},j\in\mathbb{Z},k\in\mathbb{Z}^n} \left|\langle f, \psi^\ell_{j,k}\rangle\right|^2 \tag{11}$$

for all $f \in L^2(\mathbb{R}^n)$ if and only if the set of functions $\{\psi^\ell\}_{\ell\in\mathbb{L}}$ satisfies the following two equalities:

$$\sum_{\ell\in\mathbb{L},j\in\mathbb{Z}} \left|\widehat{\psi}^\ell(2^j\xi)\right|^2 = 1, \qquad a.e.\, \xi \in \mathbb{R}^n, \tag{12}$$

$$\sum_{\ell\in\mathbb{L},j\in\mathbb{Z}_+} \widehat{\psi}^\ell(2^j\xi)\overline{\widehat{\psi}^\ell(2^j(\xi + 2\pi q))} = 0, \qquad a.e.\, \xi \in \mathbb{R}^n,\ \ \forall q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n, \tag{13}$$

where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n$ means that at least one component q_j is odd.

Corollary 1. *Under the hypotheses of Theorem 3, any function $f \in L^2(\mathbb{R}^n)$ admits the tight wavelet frame expansion*

$$f(x) = \sum_{\ell\in\mathbb{L},j\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \psi^\ell_{jk}\rangle\, \psi^\ell_{jk}(x). \tag{14}$$

By using the localization property of the frame wavelet in the Fourier domain, one can study the directions of growth of $\widehat{f}(\xi)$ by looking at the size of the frame coefficients

$$\langle f, \psi^\ell_{jk}\rangle = (2\pi)^{-n}\langle \widehat{f}, \widehat{\psi}^\ell_{jk}\rangle. \tag{15}$$

Moreover, by using the localization property of the frame wavelets in x-space, one can localize the singular support of $f(x)$ by varying ℓ, j and k in (15).

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Division of Mathematical Sciences, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

E-mail address: ashino@cc.osaka-kyoiku.ac.jp