WAVELET FRAMES AND MULTIRESOLUTION ANALYSIS

RYUICHI ASHINO, STEVEN J. DESJARDINS, MICHIHIRO NAGASE AND RÉMI VAILLANCOURT

1. Introduction

Wavelets have proven to be useful decomposition tools in a wide variety of applications throughout mathematics, science, and engineering. For example, the still-image compression standard known as JPEG2000 includes a wavelet option and the next video compression standard, MPEG-4, will be entirely wavelet based.

Hyperfunctions, which were introduced by Sato [20] and extensively developed by the Kyoto school of mathematics, can be considered to be sums of boundary values of holomorphic functions defined in infinitesimal wedges. Microlocal analysis plays an important role in the theory of hyperfunctions, partial differential operators, and many other areas. In this theory, one can define the product of distributions and discuss the partial regularity of multidimensional distributions with respect to any independent variable.

In this paper, we discuss some particular wavelet frame constructions and related multiresolution analyses which are suited for microlocal filtering. In particular, expansion of a function in terms of smooth tight frames gives a rough estimate of its microlocal content, revealing directions of analyticity. The resolution of these smooth tight wavelet frames in any given direction of analyticity can be made as fine as desired, at the cost of increasing the multiplicity of the wavelet functions.

2. Preliminaries

Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}[f] = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx$$

and the inverse Fourier transform of a function $g \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}^{-1}[g] = \check{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} g(\xi)\,d\xi.$$

The following three operators are unitary on $L^2(\mathbb{R}^n)$.

$$\begin{aligned}
&T_y f(x) = f(x-y), && \text{Translation operator,}\\
&M_\xi f(x) = e^{ix\xi} f(x), && \text{Modulation operator,}\\
&D_\rho f(x) = |\rho|^{-n/2} f(\rho^{-1}x), \quad \rho \in \mathbb{R}\backslash\{0\}, && \text{Dilation operator.}
\end{aligned}$$

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In particular, these are surjections preserving the norm of $L^2(\mathbb{R}^n)$:

$$\|T_y f\| = \|f\|, \qquad \|M_\xi f\| = \|f\|, \qquad \|D_\rho f\| = \|f\|.$$

and the adjoint operators of these three operators are given by their inverses, respectively:

$$\langle T_y f, g\rangle = \langle f, T_{-y} g\rangle, \qquad \langle M_\xi f, g\rangle = \langle f, M_{-\xi} g\rangle, \qquad \langle D_\rho f, g\rangle = \langle f, D_{1/\rho} g\rangle.$$

3. Tight Wavelet Frames

Since the dual of a tight frame is a constant multiple of the frame itself, see [1], recovering functions from their frame coefficients does not require the computation of the dual frame. Hereafter, we shall focus on tight wavelet frames.

Given $f \in L^2(\mathbb{R}^n)$, let f_{jk} denote the scaled and shifted function

$$f_{jk} := D_{1/2^j} T_k f, \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n. \tag{1}$$

Let $\mathbb{L}$ be a finite index set. A system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called a *tight wavelet frame* with frame bound A if

$$f(x) = \frac{1}{A} \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \langle f, \psi^\ell_{jk}\rangle\, \psi^\ell_{jk}(x), \qquad \forall f \in L^2(\mathbb{R}^n). \tag{2}$$

We recall that a system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called an *orthonormal wavelet basis* if it is an orthonormal basis for $L^2(\mathbb{R}^n)$. This is equivalent to saying that the system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n}$ is a tight wavelet frame with frame bound 1 and $\|\psi^\ell\|_{L^2(\mathbb{R}^n)} = 1$ for $\ell \in \mathbb{L}$.

The following general theorem which is essentially Theorem 1 stated and proved in [13] for $\mathbb{R}^n$, gives necessary and sufficient conditions to have a tight wavelet frame in $\mathbb{R}^n$ with frame bound 1.

Theorem 1. *Suppose $\psi^\ell \subset L^2(\mathbb{R}^n)$ for $\ell \in \mathbb{L}$, then*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \left|\langle f, \psi^\ell_{j,k}\rangle\right|^2 \tag{3}$$

for all $f \in L^2(\mathbb{R}^n)$ if and only if the set of functions $\{\psi^\ell\}_{\ell\in\mathbb{L}}$ satisfies the following two equalities:

$$\sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}}} \left|\widehat{\psi}^\ell(2^j\xi)\right|^2 = 1, \qquad a.e.\, \xi \in \mathbb{R}^n, \tag{4}$$

$$\sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}_+}} \widehat{\psi}^\ell(2^j\xi)\overline{\widehat{\psi}^\ell(2^j(\xi + 2\pi q))} = 0, \qquad a.e.\, \xi \in \mathbb{R}^n,\ \ \forall q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n, \tag{5}$$

where $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and $q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n$ means that at least one component q_j is odd.

Corollary 1. *Under the hypotheses of Theorem 1, any function $f \in L^2(\mathbb{R}^n)$ admits the tight wavelet frame expansion*

$$f(x) = \sum_{\substack{\ell \in \mathbb{L} \\ j \in \mathbb{Z} \\ k \in \mathbb{Z}^n}} \langle f, \psi^{\ell}_{jk} \rangle \, \psi^{\ell}_{jk}(x). \tag{6}$$

By using the localization property of the frame wavelet in the Fourier domain, one can study the directions of growth of $\widehat{f}(\xi)$ by looking at the size of the frame coefficients

$$\langle f, \psi^{\ell}_{jk} \rangle = (2\pi)^{-n} \langle \widehat{f}, \widehat{\psi}^{\ell}_{jk} \rangle. \tag{7}$$

Moreover, by using the localization property of the frame wavelets in x-space, one can localize the singular support of $f(x)$ by varying ℓ, j and k in (7).

4. Frame Multiresolution Analysis

The notion of frame multiresolution analysis was introduced by Benedetto and Li [9]. Let us recall that an *(orthonormal) multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying

(i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;
(ii) $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$;
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;
(v) There exists a function $\phi \in V_0$ such that $\{T_k\phi\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis for V_0.

The function $\phi \in L^2(\mathbb{R}^n)$ whose existence is asserted in condition (v) is called an *(orthonormal) scaling function* of the given orthonormal multiresolution analysis.

A *frame multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-1) There exists a function $\phi \in V_0$ such that $\{T_k\phi\}_{k\in\mathbb{Z}^n}$ is a frame for V_0.

The function $\phi \in L^2(\mathbb{R}^n)$ whose existence is asserted in condition (v-1) is called a *frame scaling function* of the given frame multiresolution analysis.

Let D be a finite index set. An *(orthonormal) multiresolution analysis for multiwavelets* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-2) There exists a system of functions $\{\phi_\delta\}_{\delta\in D} \subset V_0$ such that $\{T_k\phi_\delta\}_{\delta\in D,\, k\in\mathbb{Z}^n}$ is an orthonormal basis for V_0.

The set of functions $\{\phi_\delta\}_{\delta\in D}$ whose existence is asserted in condition (v-2) is called a set of *(orthonormal) multiscaling functions.*

A *frame multiresolution analysis for multiwavelets* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$, of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-3) There exists a system of functions $\{\phi_\delta\}_{\delta\in D} \subset V_0$ such that $\{T_k\phi_\delta\}_{\delta\in D,\, k\in\mathbb{Z}^n}$ is a frame for V_0.

The set of functions $\{\phi_\delta\}_{\delta\in D}$ whose existence is asserted in condition (v-3) is called a set of *frame multiscaling functions.*

5. Microlocal Analysis

Our approach to microlocal analysis is based on the theory of hyperfunctions ([16], [17], [18]). Hyperfunctions are powerful tools in several applications; for example, vortex sheets in two-dimensional fluid dynamics are a realization of hyperfunctions of one variable. Microlocal analysis deals with the direction along which a hyperfunction can be extended analytically. In other words, it decomposes the "singularity" into microlocal directions. Microlocal analysis plays an important role in the theory of hyperfunctions, partial differential operators, and other areas. In this theory, for example, one can consider the product of hyperfunctions and discuss the partial regularity of hyperfunctions with respect to any independent variable.

Here, we give only a rough sketch. A more complete treatment of microlocal filtering can be found in R. Ashino, C. Heil, M. Nagase, and R. Vaillancourt [4]. (See also [3]). The important point is to find directions in which a hyperfunction can be continued analytically. Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\Gamma \subset \mathbb{R}^n$ be a convex open cone with vertex at 0. From now on, every cone is assumed to have vertex at 0. The set $\Omega + i\Gamma \subset \mathbb{C}^n$ is called a *wedge*. An *infinitesimal wedge* $\Omega + i\Gamma 0$ is an open set $U \subset \Omega + i\Gamma$ which approaches asymptotically to Γ as the imaginary part of U tends to 0. (Figure 1.)

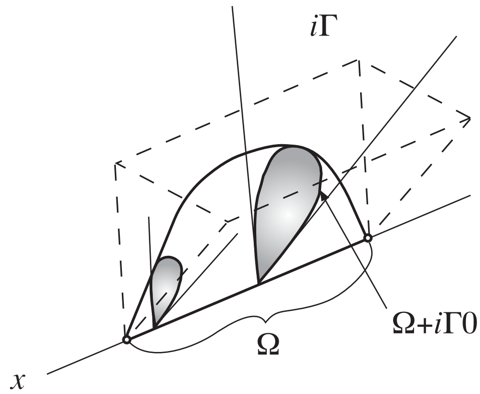

Figure 1. An infinitesimal wedge $\Omega + i\Gamma 0$.

A *hyperfunction* $f(x)$ can be defined as a sum

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \qquad x \in \Omega,$$

of formal boundary values

$$F_j(x + i\Gamma_j 0) = \lim_{\substack{y \to 0 \\ x+iy \in \Omega + i\Gamma_j 0}} F_j(x + iy)$$

of holomorphic functions $F_j(z)$ in the infinitesimal wedges $\Omega + i\Gamma_j 0$.

A hyperfunction is said to be *micro-analytic* in the direction $\xi_0 \in \mathbb{S}^{n-1}$ at $x_0 \in \mathbb{R}^n$ or in short, at (x_0, ξ_0), if there exists a neighborhood Ω of x_0 and holomorphic functions F_j in infinitesimal wedges $\Omega + i\Gamma_j 0$ such that $f = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0)$ and

$$\Gamma_j \cap \{y \in \mathbb{R}^n \ : \ y \cdot \xi_0 < 0\} \neq \emptyset$$

for all j.

A simple aspect of the relation between micro-analyticity and the Fourier transform is given as follows.

Lemma 1. *Let $\Gamma \subset \mathbb{R}^n$ be a closed cone and $x_0 \in \mathbb{R}^n$. For a tempered distribution f, if there exists a tempered distribution g such that* $\operatorname{supp} \widehat{g} \subset \Gamma$ *and $f - g$ is analytic in a neighborhood of x_0, then f is micro-analytic at (x_0, ξ) for every $\xi \in \Gamma^c \cap \mathbb{S}^{n-1}$, where Γ^c denotes the complement of Γ.*

Our aim in this paper is to answer to the following questions:

- Is it possible to construct orthonormal or tight frame multiwavelets $\Psi = \{\psi^\delta\}_{\delta \in D}$ corresponding to each microanalytic direction $\xi \in \mathbb{S}^{n-1}$?
- Is it possible to obtain information on the microlocal content of $f \in L^2(\mathbb{R}^n)$ from the wavelet coefficients $\langle f, \psi^\delta_{jk} \rangle$?
- Can orthonormal or tight frame multiwavelet filtering separate microlocal contents?

We shall construct orthonormal multiwavelet bases or tight frames which enable us to obtain information on the microlocal content of signals or functions. Since this separation of microlocal contents can be considered as a filtering operation, we call it *microlocal filtering*.

6. One-dimensional Orthonormal Microlocal Filtering

Our aim is to construct wavelets $\{\phi_\delta\}_{\delta \in D}$ having good localization both in the base space $\mathbb{R}$ and in the direction space $\mathbb{S}^0 = \{\pm 1\}$ within the limits of the uncertainty principle. Here good localization at a point $(x_0, \xi_0) \in \mathbb{R} \times \mathbb{S}^0$, which is called *good microlocalization*, means that ϕ_δ is essentially concentrated in a neighborhood of a point $x_0 \in \mathbb{R}$ and $\widehat{\phi_\delta}$ is essentially concentrated in a conic neighborhood of a point $\xi_0 \in \mathbb{S}^0$.

Define the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ by

$$H^2(\mathbb{R}_\pm) = \big\{ f \in L^2(\mathbb{R}) : \ \widehat{f}(\xi) = 0 \quad \text{a.e. } \xi \leq (\geq)\, 0 \big\}.$$

Each function of $H^2(\mathbb{R}_\pm)$ has good localization in the direction space $\mathbb{S}^0 = \{\pm 1\}$. Hence if we construct wavelets in $H^2(\mathbb{R}_\pm)$ with good localization in the base space, those wavelets have good microlocalization.

In these cases, an orthonormal wavelet function ψ_+ and an orthonormal scaling function ϕ_+ for orthonormal wavelets of $H^2(\mathbb{R}_+)$ are defined by

$$\widehat{\psi}_+ = \chi_{[2\pi, 4\pi]}, \qquad \widehat{\phi}_+ = \chi_{[0, 2\pi]}.$$

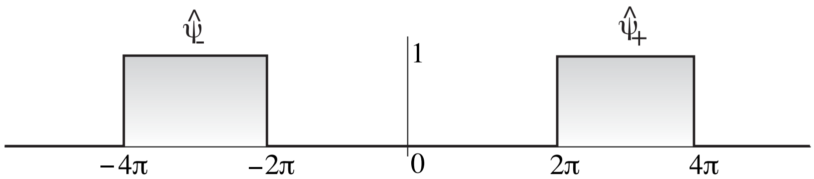

Figure 2. The Fourier transform of the orthonormal wavelet functions ψ_+ and ψ_-.

From the two-scale relation

$$2\widehat{\phi}_+(2\xi) = m_0(\xi)\widehat{\phi}_+(\xi)$$

it is found that the corresponding lowpass filter is

$$m_0(\xi) = 2\chi_{[0,\pi]}(\xi) = 2\widehat{\phi}_+(2\xi)$$

on $[0, 2\pi)$, and extended 2π-periodically. From the two-scale relation

$$2\widehat{\psi}_+(2\xi) = e^{i\xi}\,\overline{m_0(\xi+\pi)}\widehat{\phi}_+(\xi) = m_1(\xi)\widehat{\phi}_+(\xi)$$

it is found that the corresponding highpass filter is

$$m_1(\xi) = e^{i\xi}\,\overline{m_0(\xi+\pi)} = 2\widehat{\psi}_+(2\xi)$$

on $[0, 2\pi)$, and extended 2π-periodically.

By the same argument, we have an orthonormal wavelet function ψ_- and an orthonormal scaling function ϕ_- for orthonormal wavelets of $H^2(\mathbb{R}_-)$. Since

$$L^2(\mathbb{R}) = H^2(\mathbb{R}_+) \oplus H^2(\mathbb{R}_-),$$

$\{\psi_+, \psi_-\}$ and $\{\phi_+, \phi_-\}$ can be regarded as sets of orthonormal multiwavelet functions and orthonormal multiscaling functions, respectively, of $L^2(\mathbb{R})$. This decomposition of $L^2(\mathbb{R})$ into the orthogonal sum of the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ corresponds to the intuitive definition of hyperfunction:

$$f(x) = F_+(x+i0) - F_-(x-i0),$$

where $F_+(z)$ and $F_-(z)$ are holomorphic in the upper half plane and in the lower half plane, respectively.

Auscher [8] essentially proved that there is no smooth orthonormal wavelet ψ in the classical Hardy space $H^2(\mathbb{R}_+)$, that is, there is no orthonormal wavelet ψ whose Fourier transform $\widehat{\psi}$ is continuous on $\mathbb{R}$ and satisfies the regularity condition:

$$\exists\alpha > 0; \qquad |\widehat{\psi}(\xi)| = O\big((1+|\xi|)^{-\alpha-1/2}\big) \quad \text{at } \infty.$$

The decay of a function at infinity in x space corresponds to the smoothness of its Fourier transform in ξ space. Hence the non-existence of smooth wavelets implies that it is impossible to have any smooth orthonormal wavelet having good microlocalization. Thus our aim is to construct smooth tight frame wavelets with good microlocalization properties.

7. Multi-dimensional Orthonormal Microlocal Filtering

The following notation will be used.

- $\eta = (\eta_1, \dots, \eta_n) \in H := \{\pm 1\}^n$.
- $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in E := \{0, 1\}^n \setminus \{0\}, \quad j \in \mathbb{Z}_+$.
- $Q_\eta := \prod_{k=1}^n [0, \eta_k], \quad \varepsilon.*\eta := (\varepsilon_1\eta_1, \dots, \varepsilon_n\eta_n)$.
- $\mathcal{Q}_{j,\varepsilon,\eta} := \big\{\prod_{k=1}^n [\eta_k(\ell_k - 1), \eta_k\ell_k] + 2^j(\varepsilon.*\eta) : 1 \le \ell_1, \dots, \ell_n \le 2^j,\ \ell_1, \dots, \ell_n \in \mathbb{N}\big\}$.
- $\mathbb{Z}_+^{E\times H}$ is the set of all functions from $E \times H$ to $\mathbb{Z}_+$.

Theorem 2. *Fix $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$. For a cube $Q \in \mathcal{Q}_{j,\varepsilon,\eta}$, define ψ_Q by*

$$\widehat{\psi}_Q = \chi_{2\pi Q},$$

where $\chi_{2\pi Q}$ is the characteristic function of the cube $2\pi Q$. For $\rho \in \mathbb{Z}_+^{E\times H}$, let

$$\mathcal{Q}_\rho := \bigcup_{(\varepsilon,\eta)\in E\times H} \mathcal{Q}_{\rho(\varepsilon,\eta),\varepsilon,\eta}.$$

Then $\Psi := \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ is a set of orthonormal wavelets. Define ϕ_η by

$$\widehat{\phi}_\eta := \chi_{2\pi Q_\eta}.$$

Then $\{\phi_\eta\}_{\eta\in H}$ is a set of frame scaling functions for these wavelets.

In particular, when $\rho(\varepsilon,\eta)$ is constant, Ψ is a set of multiwavelets.

Figure 3 illustrates the 2-D multiwavelets constructed in Theorem 2. Multiwavelets are masks in Fourier space — they are characteristic functions of cubes $2\pi Q$. The left part of Fig. 3 shows 12 multiwavelet functions. For finer resolution in Fourier space, we need a greater number of multiwavelets. The right part of Fig. 3 shows 27 multiwavelet functions.

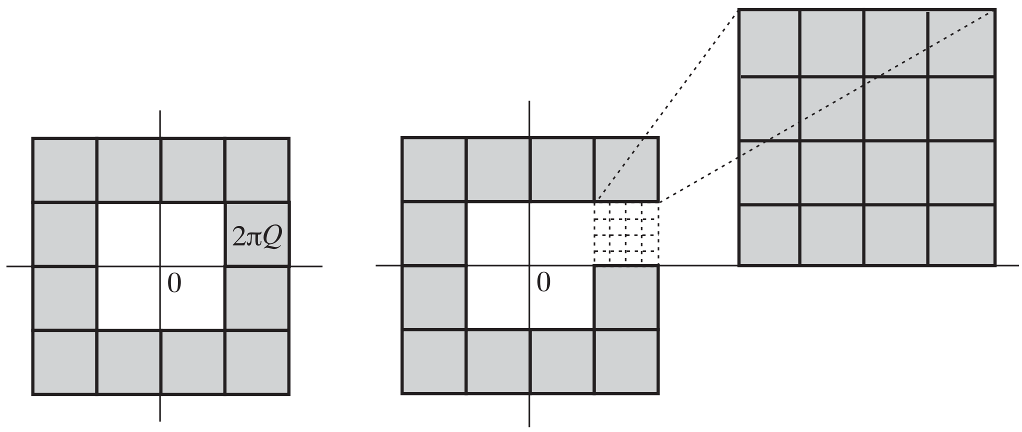

FIGURE 3. 2-D orthonormal multiwavelet functions in Fourier space.

8. Multi-dimensional Frame Microlocal Filtering

Smooth tight multiwavelet frames are obtained by convolving characteristic functions of cubes πQ so that the support of the smoothed functions have support inside cubes $2\pi Q$. This is achieved by considering the next inside annulus of cubes πQ in the left part of Fig. 3.

Let $\vartheta(t)$ be a $C_0^\infty(\mathbb{R})$-function of one variable satisfying

$$\vartheta(t) \geq 0, \qquad \vartheta(t) = \vartheta(-t), \qquad \int_{\mathbb{R}} \vartheta(t)\,dt = 1, \qquad \vartheta(t) = \begin{cases} 1, & |t| \leq \frac{1}{3}; \\ 0, & |t| \geq \frac{2}{3}. \end{cases}$$

For $\alpha > 0$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, let

$$\vartheta_\alpha(\xi) = \frac{1}{\alpha^n} \prod_{j=1}^{n} \vartheta\Big(\frac{\xi_j}{\alpha}\Big).$$

We have the following theorem.

Theorem 3. *Fix $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$, and $\alpha \in (0, 1/2)$. Define*

$$\lambda_Q(\xi) := (\vartheta_\alpha * \chi_{\pi Q})(\xi) = \int_{\mathbb{R}^n} \vartheta_\alpha(\xi - \zeta)\, \chi_{\pi Q}(\zeta)\, d\zeta, \qquad Q \in \mathcal{Q}_{j,\varepsilon,\eta},$$

where $\chi_{\pi Q}$ is the characteristic function of the cube πQ. For $\rho \in \mathbb{Z}_+^{E\times H}$, let

$$\tau_\rho(\xi) := \sum_{j\in\mathbb{Z}, Q\in\mathcal{Q}_\rho} |\lambda_Q(2^j\xi)|^2,$$

and, for $Q \in \mathcal{Q}_\rho$, define $\psi_Q(x)$ by

$$\widehat{\psi}_Q(\xi) := \tau_\rho(\xi)^{-1/2}\, \lambda_Q(\xi).$$

Then $\Psi := \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ is a set of tight frame wavelets.

Theorem 3 follows from Theorem 1.

9. Numerical Restoration of Images

In this section, we apply the above theory to the restoration of finite images represented by matrices. Since the Fourier transform of a finite region gives rise to oscillations of the type of cardinal sine, care must be taken in the restoration process.

The restoration process involves the following steps.

- The figure A to be restored is Fourier transformed into B.
- B is filtered by multiplication with a tapered characteristic function with support far from the origin and at right angle with the singularity to be localized. This produces C.
- In view of the Plancherel theorem, the wavelet coefficients of C, in (7),
$$\langle \widehat{f}, \widehat{\psi}^\ell_{jk}\rangle = (2\pi)^2 \langle f, \psi^\ell_{jk}\rangle,$$
are constructed in the Fourier domain and used in the x domain, to produce D which is the wavelet frame expansion (6) of Corollary 1.
- The extra width of D, caused by the side lobes in the support of ψ^ℓ_{jk}, is narrowed to eliminate oscillations due the cardinal sine effect when transforming functions with finite support.
- A tuned multiple of D is subtracted from A to restore the original image.

In Fig. 4, the scarred woman image is restored. One notices in the top right part of the figure the wide width of the negative of the Fourier transform of the one-bit wide short scar. The frame expansion of the inverse Fourier transform of the top right part produced a five-bit wide segment. The width of this segment was reduced to one bit shown as a negative in the bottom left part of the figure. A multiple of the bottom left part of the figure, as a positive, was subtracted from the top left part to produce the restored woman figure shown in the bottom right part. In this case, only one frame wavelet was used as highpass filter in the top right part of the figure in the Fourier domain. Using a second filter in the lower left part of the Fourier domain does not seem to modify the final result.

In Fig. 5, the boy image with a diagonal line is restored. One notices in the top right part of the figure the narrow width of the negative of the Fourier transform of the

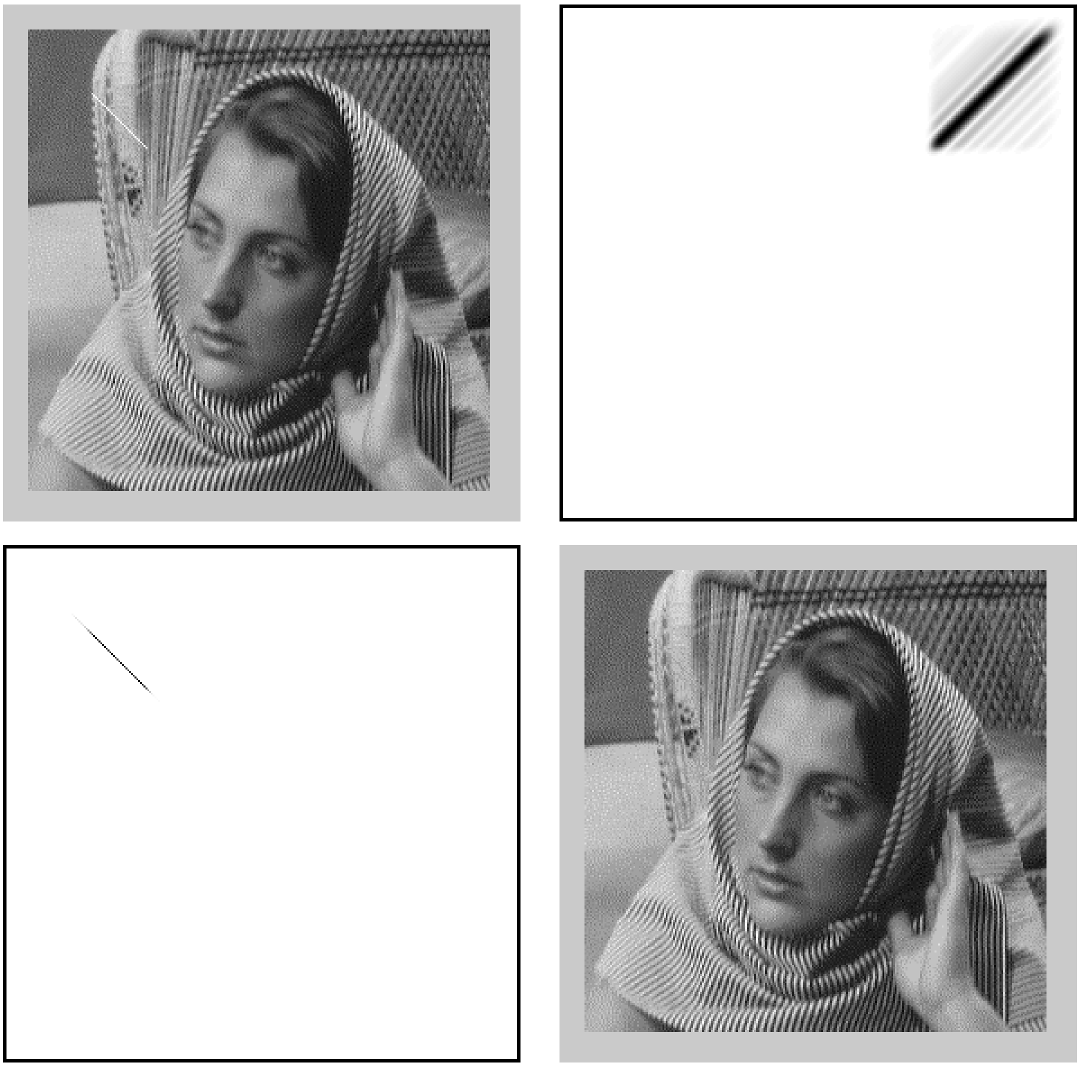

FIGURE 4. Top left: positive scarred woman figure. Top right: framed negative filtered Fourier transform of top left figure. Bottom left: framed negative frame expansion of the inverse Fourier transform of top right figure. Bottom right: positive restored woman figure.

one-bit wide long diagonal line. The frame expansion of the inverse Fourier transform of the top right part produced an eight-bit wide segment. The width of this segment was reduced to one bit. Moreover, fine tuning required that the fourth root of this segment be taken. The result is shown as a negative in the bottom left part of the figure. A multiple of the bottom left part of the figure, as a positive, was subtracted from the top left part to produce the restored boy figure shown in the bottom right part. In this case, two frame wavelets were used as highpass filters in the top right and bottom left parts of the figure in the Fourier domain. Using only one filter in the upper right or lower left part in the Fourier domain does not seem to modify the final result.

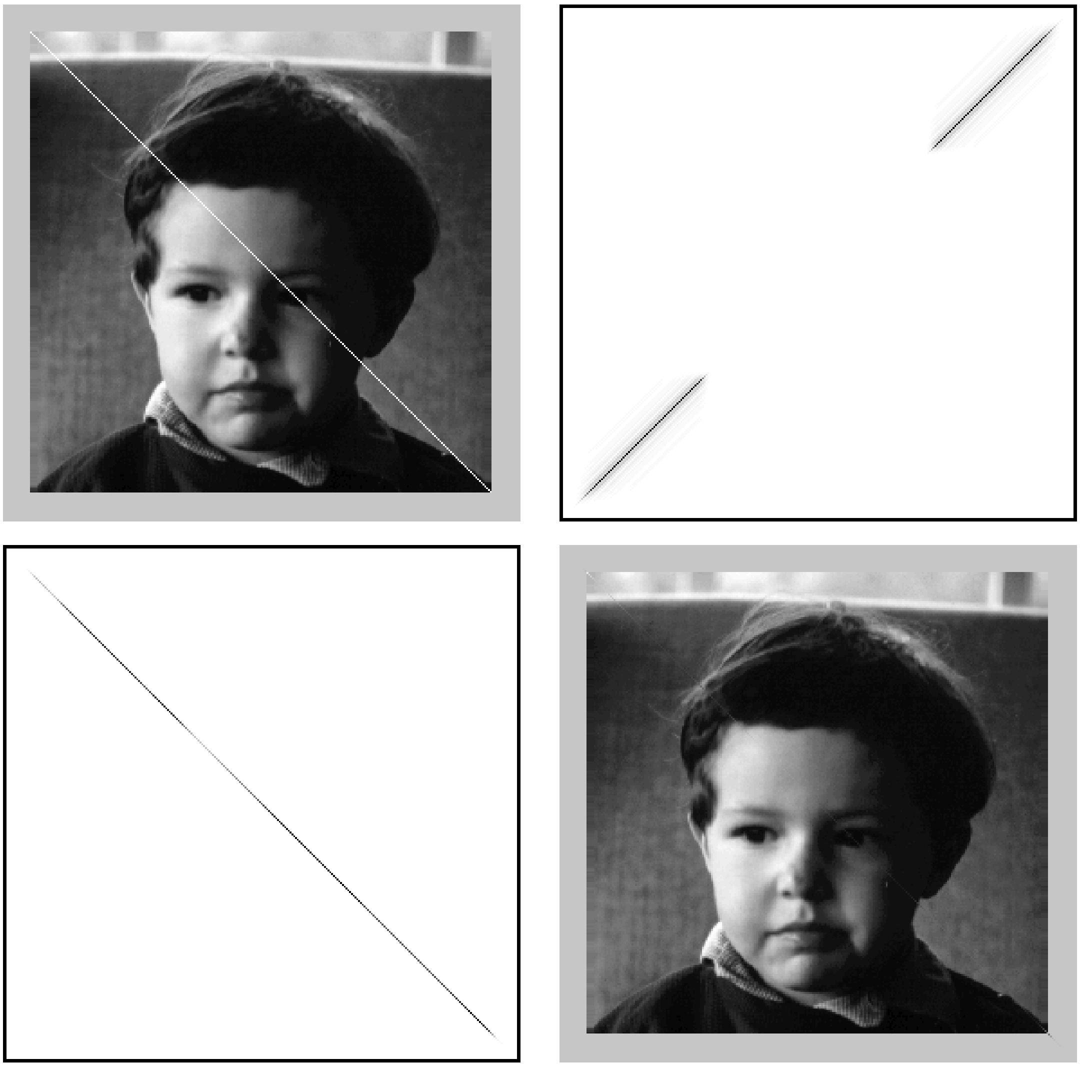

Figure 5. Top left: positive boy figure with diagonal line. Top right: framed negative filtered Fourier transform of top left figure. Bottom left: framed negative frame expansion of the inverse Fourier transform of top right figure. Bottom right: positive restored boy figure.

References

[1] R. Ashino, *Some topics on wavelets*, Seminar Notes of Mathematical Sciences, Ibaraki University **6** (2003) to appear.

[2] R. Ashino, S. J. Desjardins, C. Heil, M. Nagase, and R. Vaillancourt, *Microlocal analysis, smooth frames and denoising in Fourier space*, Asian Information-Science-Life, An International Journal **1** (2002) to appear.

[3] R. Ashino, S. J. Desjardins, C. Heil, M. Nagase, and R. Vaillancourt, *Smooth tight frame wavelets and image microanalysis in the Fourier domain*, Comput. Math. Appl., to appear.

[4] R. Ashino, C. Heil, M. Nagase and R. Vaillancourt, *Microfiltering with multiwavelets*, Computers Math. Applic. **41**(1-2) (2001) 111–133.

[5] R. Ashino, C. Heil, M. Nagase and R. Vaillancourt, *Microlocal analysis and multiwavelets*, in Geometry, Analysis and Applications (Varanasi 2000), R. S. Pathak ed., World Scientific, River Edge NJ, 2001, pp. 293–302.

[6] R. Ashino, C. Heil, M. Nagase et R. Vaillancourt, *Multiwavelets, pseudodifferential operators and microlocal analysis*, in Wavelets Analysis and Applications, D. Deng, D. Huang, R.-Q. Jia, and W. Lin, eds., pp. 9–20, AMS/IP Studies in Advanced Mathematics, Vol. 25, 2002.

[7] R. Ashino and T. Mandai, *Wavelet Bases for Microlocal Filtering and the Sampling Theorem in* $L_p(\mathbb{R}^n)$, Applicable Analysis, to appear.

[8] P. Auscher, *Il n'existe pas de bases d'ondelettes régulières dans l'espace de Hardy* H^2, C. R. Acad. Sci. Paris, **315**, Série I (1992), 769-772.

[9] J. J. Benedetto and S. Li, *The theory of multiresolution analysis and applications to filter banks*, Applied Computational Harm. Anal. **5** (1998) 389–427.

[10] I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia PA, 1992.

[11] S. J. Desjardins, *Image analysis in Fourier space*, Doctoral dissertation, University of Ottawa, Ottawa, Canada, 2002.

[12] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952) 341–366.

[13] M. Frazier, G. Garrigós, K. Wang and G. Weiss, *A characterization of functions that generate wavelet and related expansion*, in Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), J. Fourier Anal. Appl. **3** (1997), 883–906.

[14] C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review, **31**(4) (1989), pp. 628–666.

[15] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, FL, 1996, Chapters 7 and 8.

[16] A. Kaneko, *Microlocal analysis*, in Encyclopedia of Mathematics, Kluwer Academic Publishers, Dordrecht, 1990.

[17] A. Kaneko *Introduction to hyperfunctions*, Kluwer Academic Publishers, Dordrecht, 1988.

[18] A. Kaneko *Linear partial differential equations with constant coefficients*, Iwanami, Tokyo, 1992 (Japanese).

[19] S. Mallat *A wavelet tour of signal processing*, 2nd edition, Academic Press, San Diego CA, 1999.

[20] M. Sato, *Theory of hyperfunctions I*, J. Fac. Sci. Univ. Tokyo, Sec. I, **8**(1) (1959), pp. 139–193.

R.A., Division of Mathematical Sciences, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

E-mail address: `ashino@cc.osaka-kyoiku.ac.jp`

M.N., Department of Mathematics, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `nagase@math.wani.osaka-u.ac.jp`

S.J.D. and R.V. Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5

E-mail address: `remi@uottawa.ca`